

TWISTING OF THE QUANTUM DOUBLE AND THE WEYL ALGEBRA

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ABSTRACT. Quantum double construction, originally due to Drinfeld and has been since generalized even to the operator algebra framework, is naturally associated with a certain (quasitriangular) R -matrix \mathcal{R} . It turns out that \mathcal{R} determines a twisting of the comultiplication on the quantum double. It then suggests a twisting of the algebra structure on the dual of the quantum double. For $D(G)$, the C^* -algebraic quantum double of an ordinary group G , the “twisted $\widehat{D(G)}$ ” turns out to be the Weyl algebra $C_0(G) \times_\tau G$, which is in turn isomorphic to $\mathcal{K}(L^2(G))$. This is the C^* -algebraic counterpart to an earlier (finite-dimensional) result by Lu. It is not so easy technically to extend this program to the general locally compact quantum group case, but we propose here some possible approaches, using the notion of the (generalized) Fourier transform.

1. INTRODUCTION

There are a few different approaches to formulate the notion of quantum groups, which are generalizations of ordinary groups. In the finite-dimensional case, they usually come down to Hopf algebras [1], [14], although there actually exist examples of quantum groups that cannot be described only by Hopf algebra languages. More generally, the approaches to quantum groups include the (purely algebraic) setting of “quantized universal enveloping (QUE) algebras” [6], [4]; the setting of multiplier Hopf algebras and algebraic quantum groups [19], [9]; and the (C^* - or von Neumann algebraic) setting of locally compact quantum groups [10], [11], [13], [20]. In this paper, we are mostly concerned with the setting of C^* -algebraic locally compact quantum groups.

In all these approaches to quantum groups, one important aspect is that the category of quantum groups is a “self-dual” category, which is not the case for the (smaller) category of ordinary groups. To be more specific, a typical quantum group A is associated with a certain dual object \hat{A} , which is also a quantum group, and the dual object, $\hat{\hat{A}}$, of the dual quantum group is actually isomorphic to A . This result, $\hat{\hat{A}} \cong A$, is a generalization of the Pontryagin duality, which holds in the smaller category of abelian locally compact groups.

For a finite dimensional Hopf algebra H , its dual object is none other than the dual vector space H' , with its Hopf algebra structure obtained naturally

from that of H . In general, however, a typical quantum group A would be infinite dimensional, and in that case, the dual vector space is too big to be given any reasonable structure (For instance, one of the many drawbacks is that $(A \otimes A)'$ is strictly larger than $A' \otimes A'$).

In each of the approaches to quantum groups, therefore, a careful attention should be given to making sense of what the dual object is for a quantum group, as well as to exploring the relationship between them. This is especially true for the analytical settings, where the quantum groups are required to have additional, topological structure. The success of the locally compact quantum group framework by Kustermans and Vaes [10], and also by Masuda, Nakagami, and Woronowicz [13] is that they achieve the definition of locally compact quantum groups so that it has the self-dual property.

Meanwhile, given a Hopf algebra H and its dual \hat{H} , there exists the notion of the “quantum double” $H_D = \hat{H}^{\text{op}} \bowtie H$ (see [6], [14]). This notion can be generalized even to the setting of locally compact quantum groups: From a von Neumann algebraic quantum group (N, Δ) , one can construct the quantum double (N_D, Δ_D) . See Section 2 below.

The quantum double is associated with a certain “quantum universal R -matrix” type operator $\mathcal{R} \in N_D \otimes N_D$. It turns out that \mathcal{R} determines a left cocycle for Δ_D , and allows us to *twist (or deform)* the comultiplication on N_D , or its C^* -algebraic counterpart A_D . The result, $(A_D, \mathcal{R}\Delta_D)$, can no longer become a locally compact quantum group, but it suggests a twisting of the algebra structure at the level of $\widehat{A_D}$, the dual of the quantum double. Our intention here is to explore this algebra, the “deformed $\widehat{A_D}$ ”.

There are two crucial obstacles in carrying out this program. For one thing, the C^* -algebra $\widehat{A_D}$ itself can be rather complicated in general. In addition, unlike in the algebraic approaches, even the simple tool like the dual pairing is not quite easy to work with. In the locally compact quantum group framework, the dual pairing between a quantum group A and its dual \hat{A} is defined at dense subalgebra level, by using the multiplicative unitary operator associated with A and \hat{A} . While it is a correct definition (in the sense that it is a natural generalization of the obvious dual pairing between H and H' in the finite-dimensional case), the way it is defined makes it rather difficult to work with. For instance, there is no straightforward way of obtaining a dual object of a C^* -bialgebra.

These technical difficulties cannot be totally overcome, but we can improve the situation by having a better understanding of the duality picture. Recently in [8], motivated by Van Daele’s work in the multiplier Hopf algebra framework [21], the author defined the (generalized) Fourier transform between a locally compact quantum group and its dual. In addition, an alternative description of the dual pairing was found (see Section 4 of [8]), in terms of the Haar weights and the Fourier transform. This alternative perspective to the dual pairing is useful in our paper.

In the case of an ordinary locally compact group G , so for $A = C_{red}^*(G)$ (the “reduced group C^* -algebra”) and $\hat{A} = C_0(G)$, the quantum double turns out to be $A_D = C_0(G) \rtimes_\alpha G$, the crossed product C^* -algebra given by the group G acting on itself by conjugation α . It is also known that $\widehat{A_D} = C_{red}^*(G) \otimes C_0(G)$. After carrying out the twisting process of $\widehat{A_D}$ as described above, we will see in Section 5 below that it gives rise to the crossed product C^* -algebra $B = C_0(G) \rtimes_\tau G$, where τ is the translation. This algebra is often called the “Weyl algebra”. It is quite interesting to observe this relationship between the quantum double (a quantum group) and the Weyl algebra (no longer a quantum group), which are both well-known to appear in some physics applications.

Meanwhile, it is known that as a C^* -algebra, the Weyl algebra is isomorphic to the algebra of compact operators: $C_0(G) \rtimes_\tau G \cong \mathcal{K}(L^2(G))$. In the (finite-dimensional) Hopf algebra setting, a similar process was carried out by Lu [12], [14]: Lu’s result says that the twisting of the dual of the quantum double turns out to be isomorphic to the smash product $H \# \hat{H}$, which is in turn known to be isomorphic to $\text{End}(H)$. In this sense, our observation here will be the C^* -algebraic counterpart to Lu’s result. See also, [5], where the result is obtained in the setting of multiplier Hopf algebras.

Motivated by the results in these “good” cases, we then try to consider the case of general locally compact quantum groups. While there are technical obstacles, we propose in Section 6 a workable approach based on the property of the Fourier transform. For a general (not necessarily regular) locally compact quantum group A , the C^* -algebra of “deformed $\widehat{A_D}$ ” may no longer be isomorphic to $\mathcal{K}(\mathcal{H})$ and can be quite complicated: It may not even be of type I.

Here is how the paper is prepared: In Section 2, we give basic definitions and review some results about locally compact quantum groups and its dual. We will also describe the dual pairing map, including an alternative characterization obtained recently by the author.

In Section 3, we will discuss the quantum double construction. This is a special case of the “double crossed product” construction developed by Baaj and Vaes in [3]. However, the scope of that paper is a little too general, and we needed to have an explicit summary written out on the quantum double construction for a general locally compact quantum group. Some of the results here, while straightforward, were just barely noted in [3] and have not appeared elsewhere: Among such results is the discussion on the “quantum R -matrix” type operator. In Section 4, we will see how the R -matrix \mathcal{R} determines a left twisting of the comultiplication on the quantum double. It will suggest a twisting (deformation) at the dual level.

In Section 5, we consider the case of an ordinary group and its quantum double $D(G)$, then carry out the twisting of $\widehat{D(G)}$. As noted above, the result is shown to be isomorphic to the Weyl algebra. In Section 6, we consider the general case. Using the case of $D(G)$ and $\widehat{D(G)}$ as a basis, we

will collect some information that can be used in our efforts to go further into the case of general locally compact quantum groups. We will propose here a reasonable description for the deformed $\widehat{A_D}$. The notion of the generalized Fourier transform defined in [8] will play a central role.

2. PRELIMINARIES

2.1. Locally compact quantum groups. Let us first begin with the definition of a *von Neumann algebraic locally compact quantum group*, given by Kustermans and Vaes [11]. This definition is known to be equivalent to the definition in the C^* -algebra setting [10], and also to the formulation given by Masuda–Nakagami–Woronowicz [13]. Refer also to the recent paper by Van Daele [20]. We note that the existence of the Haar (invariant) weights has to be assumed as a part of the definition.

Definition 2.1. Let M be a von Neumann algebra, together with a unital normal $*$ -homomorphism $\Delta : M \rightarrow M \otimes M$ satisfying the “coassociativity” condition: $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$. Assume further the existence of a left invariant weight and a right invariant weight, as follows:

- φ is an n.s.f. weight on M that is left invariant:

$$\varphi((\omega \otimes \text{id})(\Delta x)) = \omega(1)\varphi(x), \quad \text{for all } x \in \mathfrak{M}_\varphi^+ \text{ and } \omega \in M_*^+.$$
- ψ is an n.s.f. weight on M that is right invariant:

$$\psi((\text{id} \otimes \omega)(\Delta x)) = \omega(1)\psi(x), \quad \text{for all } x \in \mathfrak{M}_\psi^+ \text{ and } \omega \in M_*^+.$$

Then we say that (M, Δ) is a *von Neumann algebraic quantum group*.

Remark. We are using the standard notations and terminologies from the theory of weights. For instance, an “n.s.f. weight” is a normal, semi-finite, faithful weight. For an n.s.f. weight φ , we write $x \in \mathfrak{M}_\varphi^+$ to mean $x \in M^+$ so that $\varphi(x) < \infty$, while $x \in \mathfrak{N}_\varphi$ means $x \in M$ so that $\varphi(x^*x) < \infty$. See [17]. Meanwhile, it can be shown that the Haar weights φ and ψ above are unique, up to scalar multiplication.

Let us fix φ . Then by means of the GNS construction $(\mathcal{H}, \iota, \Lambda)$ for φ , we may as well regard M as a subalgebra of the operator algebra $\mathcal{B}(\mathcal{H})$, such as $M = \iota(M) \subseteq \mathcal{B}(\mathcal{H})$. Thus we will have: $\langle \Lambda(x), \Lambda(y) \rangle = \varphi(y^*x)$ for $x, y \in \mathfrak{N}_\varphi$, and $a\Lambda(y) = \iota(a)\Lambda(y) = \Lambda(ay)$ for $y \in \mathfrak{N}_\varphi$, $a \in M$. Consider next the operator T , which is the closure of the map $\Lambda(x) \mapsto \Lambda(x^*)$ for $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$. Expressing its polar decomposition as $T = J\nabla^{1/2}$, we obtain in this way the “modular operator” ∇ and the “modular conjugation” J . The operator ∇ determines the modular automorphism group. Refer to the standard weight theory [17].

Meanwhile, there exists a unitary operator $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, called the *multiplicative unitary operator* for (M, Δ) . It is defined by $W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)((\Delta y)(x \otimes 1))$, for $x, y \in \mathfrak{N}_\varphi$. It satisfies the pentagon equation of Baaj and Skandalis [2]: $W_{12}W_{13}W_{23} = W_{23}W_{12}$. We also have:

$\Delta a = W^*(1 \otimes a)W$, for $a \in M$. The operator W is the “left regular representation”, and it provides the following useful characterization of M :

$$M = \overline{\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w (\subseteq \mathcal{B}(\mathcal{H})),$$

where $-^w$ denotes the von Neumann algebra closure (for instance, the closure under σ -weak topology).

If we wish to consider the quantum group in the C^* -algebra setting, we just need to take the norm completion instead, and restrict Δ to A . See [10], [20]. Namely,

$$A = \overline{\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\parallel} (\subseteq \mathcal{B}(\mathcal{H})).$$

Constructing the antipode is rather technical (it uses the right Haar weight), and we refer the reader to the main papers [10], [11]. See also an improved treatment given in [20], where the antipode is defined in a more natural way by means of Tomita–Takesaki theory. For our purposes, we will just mention the following useful characterization of the antipode S :

$$S((\text{id} \otimes \omega)(W)) = (\text{id} \otimes \omega)(W^*). \quad (2.1)$$

In fact, the subspace consisting of the elements $(\text{id} \otimes \omega)(W)$, for $\omega \in \mathcal{B}(\mathcal{H})_*$, is dense in M and forms a core for S . Meanwhile, there exist a unique $*$ -antiautomorphism R (called the “unitary antipode”) and a unique continuous one parameter group τ on M (called the “scaling group”) such that we have: $S = R\tau_{-\frac{i}{2}}$. Since $(R \otimes R)\Delta = \Delta^{\text{cop}}R$, where Δ^{cop} is the co-opposite comultiplication (i.e. $\Delta^{\text{cop}} = \chi \circ \Delta$, for χ the flip map on $M \otimes M$), the weight $\varphi \circ R$ is right invariant. So we can, without loss of generality, choose ψ to equal $\varphi \circ R$. The GNS map for ψ will be written as Γ .

From the right Haar weight ψ , we can find another multiplicative unitary V , defined by $V(\Gamma(x) \otimes \Gamma(y)) = (\Gamma \otimes \Gamma)(\Delta x)(1 \otimes y)$, for $x, y \in \mathfrak{N}_\psi$. It is the “right regular representation”, and it provides an alternative characterization of M : That is, $M = \overline{\{(\omega \otimes \text{id})(V) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w (\subseteq \mathcal{B}(\mathcal{H}))$.

Next, let us consider the *dual quantum group*. Working with the other leg of the multiplicative unitary operator W , we define:

$$\hat{M} = \overline{\{(\omega \otimes \text{id})(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w (\subseteq \mathcal{B}(\mathcal{H})).$$

This is indeed shown to be a von Neumann algebra. We can define a comultiplication on it, by $\hat{\Delta}(y) = \Sigma W(y \otimes 1)W^*\Sigma$, for all $y \in \hat{M}$. Here, Σ is the flip map on $\mathcal{H} \otimes \mathcal{H}$, and defining the dual comultiplication in this way makes it “flipped”, unlike in the purely algebraic settings (See the remark following Proposition 2.2 for more discussion.). This is done for technical reasons, so that it is simpler to work with the multiplicative unitary operator.

The general theory assures that $(\hat{M}, \hat{\Delta})$ is again a von Neumann algebraic quantum group, together with appropriate Haar weights $\hat{\varphi}$ and $\hat{\psi}$. By taking the norm completion, we can consider the C^* -algebraic quantum group $(\hat{A}, \hat{\Delta})$. The operator $\hat{W} = \Sigma W^*\Sigma$ is easily seen to be the multiplicative unitary for $(\hat{M}, \hat{\Delta})$. It turns out that $W \in M \otimes \hat{M}$ and $\hat{W} \in \hat{M} \otimes M$.

The left Haar weight $\hat{\varphi}$ on $(\hat{M}, \hat{\Delta})$ is characterized by the GNS map $\hat{\Lambda} : \mathfrak{N}_{\hat{\varphi}} \rightarrow \mathcal{H}$, which is given by the following (See Proposition 8.14 of [10]):

$$\langle \hat{\Lambda}((\omega \otimes \text{id})(W)), \Lambda(x) \rangle = \omega(x^*). \quad (2.2)$$

For this formula to make sense, we need $\omega \in \mathcal{B}(\mathcal{H})_*$ to have $L \geq 0$ such that $|\omega(x^*)| \leq L\|\Lambda(x)\|$ for all $x \in \mathfrak{N}_{\hat{\varphi}}$. It is known that for such linear forms ω , the elements $(\omega \otimes \text{id})(W)$ form a core for $\hat{\Lambda}$. See [10], [11].

The other structure maps for $(\hat{M}, \hat{\Delta})$ are defined as before, including the modular operator $\hat{\nabla}$, the modular conjugation \hat{J} , and the antipode \hat{S} . As for the antipode map \hat{S} , a similar characterization as in equation (2.1) exists, with $\hat{W} = \Sigma W^* \Sigma$ now being the multiplicative unitary. Namely, $\hat{S}((\omega \otimes \text{id})(W^*)) = (\omega \otimes \text{id})(W)$. The unitary antipode and the scaling group can be also found, giving us the polar decomposition $\hat{S} = \hat{R}\hat{\tau}_{\frac{i}{2}}$.

The modular conjugations J and \hat{J} are closely related with the antipode maps. In fact, it is known that $R(x) = \hat{J}x^*\hat{J}$, for $x \in M$ and $\hat{R}(y) = \hat{J}y^*J$, for $y \in \hat{M}$. It is also known that $\hat{J}J = \nu^{i/4}JJ\hat{J}$ (where ν is the ‘‘scaling constant’’), and that $W^* = (\hat{J} \otimes J)W(\hat{J} \otimes J)$, and $V = (\hat{J} \otimes \hat{J})\Sigma W^* \Sigma(\hat{J} \otimes \hat{J})$. We have: $V \in \hat{M}' \otimes M$, where \hat{M}' is the commutant of \hat{M} , with the opposite product. See [11] and [18], for further results on the relationships between various operators.

Repeating the whole process again, we can also construct the dual $(\hat{M}, \hat{\Delta})$ of $(\hat{M}, \hat{\Delta})$. An important result is the *generalized Pontryagin duality*, which says that $(\hat{M}, \hat{\Delta}) \cong (M, \Delta)$.

We wrap up the subsection here. For further details, we refer the reader to the fundamental papers on the subject: [2], [22], [10], [11], [13], [20].

2.2. The dual pairing. Suppose we have a mutually dual pair of quantum groups (M, Δ) and $(\hat{M}, \hat{\Delta})$. Let W be the associated multiplicative unitary operator. The dual pairing exists between M and \hat{M} , but unlike in the (purely algebraic) cases of finite-dimensional Hopf algebras or multiplier Hopf algebras, the pairing map is defined only at the level of certain dense subalgebras of M and \hat{M} . To be more specific, consider the subsets \mathcal{A} ($\subseteq M$) and $\hat{\mathcal{A}}$ ($\subseteq \hat{M}$), defined by

$$\mathcal{A} = \{(\text{id} \otimes \omega)(W) : \omega \in M_*\}$$

and

$$\hat{\mathcal{A}} = \{(\omega' \otimes \text{id})(W) : \omega' \in \hat{M}_*\}.$$

By the general theory, it is known (see [2], [11]) that the spaces \mathcal{A} and $\hat{\mathcal{A}}$ are actually (dense) subalgebras of M and \hat{M} . The dual pairing exists between $\hat{\mathcal{A}}$ and \mathcal{A} : That is, for $b = (\omega \otimes \text{id})(W) \in \hat{\mathcal{A}}$ and $a = (\text{id} \otimes \theta)(W) \in \mathcal{A}$, we have:

$$\langle b | a \rangle = \langle (\omega \otimes \text{id})(W) | (\text{id} \otimes \theta)(W) \rangle := (\omega \otimes \theta)(W) = \omega(a) = \theta(b). \quad (2.3)$$

This definition is suggested by [2]. The properties of this pairing map is given below:

Proposition 2.2. *Let (M, Δ) and $(\hat{M}, \hat{\Delta})$ be the dual pair of locally compact quantum groups, and let \mathcal{A} and $\hat{\mathcal{A}}$ be their dense subalgebras, as defined above. Then the map $\langle \cdot | \cdot \rangle : \hat{\mathcal{A}} \times \mathcal{A} \rightarrow \mathbb{C}$, given by equation (2.3), is a valid dual pairing. Moreover, we have:*

- (1) $\langle b_1 b_2 | a \rangle = \langle b_1 \otimes b_2 | \Delta(a) \rangle$, for $a \in \mathcal{A}$, $b_1, b_2 \in \hat{\mathcal{A}}$.
- (2) $\langle b | a_1 a_2 \rangle = \langle \hat{\Delta}^{\text{cop}}(b) | a_1 \otimes a_2 \rangle$, for $a_1, a_2 \in \mathcal{A}$, $b \in \hat{\mathcal{A}}$.
- (3) $\langle b | S(a) \rangle = \langle \hat{S}^{-1}(b) | a \rangle$, for $a \in \mathcal{A}$, $b \in \hat{\mathcal{A}}$.

Remark. Bilinearity of $\langle \cdot | \cdot \rangle$ is obvious, and the proof of the three properties is straightforward. See, for instance, Proposition 4.2 of [8]. Except for the appearance of the co-opposite comultiplication $\hat{\Delta}^{\text{cop}}$ in (2), the proposition shows that $\langle \cdot | \cdot \rangle$ is a suitable dual pairing map that generalizes the pairing map on (finite-dimensional) Hopf algebras. The difference is that in purely algebraic frameworks (Hopf algebras, QUE algebras, or even multiplier Hopf algebras), the dual comultiplication on H' is simply defined by dualizing the product on H via the natural pairing map between H and H' . Whereas in our case, the pairing is best defined using the multiplicative unitary operator. It turns out that defining as we have done the dual comultiplication as “flipped” makes things to become technically simpler, even with (2) causing minor annoyance.

Meanwhile, let us quote below an alternative description given in [8] of this pairing map, using the Haar weights and the generalized Fourier transform. The new descriptions are only valid on certain subspaces $D \subseteq \mathcal{A}$ and $\hat{D} \subseteq \hat{\mathcal{A}}$, but D and \hat{D} are dense subalgebras in M and \hat{M} respectively, and form cores for the antipode maps S and \hat{S} .

Theorem 2.3. *Let $D \subseteq \mathcal{A}$ and $\hat{D} \subseteq \hat{\mathcal{A}}$ be the dense subalgebras as defined in Section 4 of [8]. Then:*

- (1) *For $a \in D$, its Fourier transform is defined by*

$$\mathcal{F}(a) := (\varphi \otimes \text{id})(W(a \otimes 1)).$$

- (2) *For $b \in \hat{D}$, the inverse Fourier transform is defined by*

$$\mathcal{F}^{-1}(b) := (\text{id} \otimes \hat{\varphi})(W^*(1 \otimes b)).$$

- (3) *The dual pairing map $\langle \cdot | \cdot \rangle : \hat{\mathcal{A}} \times \mathcal{A} \rightarrow \mathbb{C}$ given in Proposition 2.2 takes the following form, if we restrict it to the level of D and \hat{D} :*

$$\begin{aligned} \langle b | a \rangle &= \langle \hat{\Lambda}(b), \Lambda(a^*) \rangle = \varphi(a \mathcal{F}^{-1}(b)) = \hat{\varphi}(\mathcal{F}(a^*)^* b) \\ &= (\varphi \otimes \hat{\varphi})[(a \otimes 1) W^*(1 \otimes b)]. \end{aligned}$$

Remark. Here, φ and $\hat{\varphi}$ are the left invariant Haar weights for (M, Δ) and $(\hat{M}, \hat{\Delta})$, while Λ and $\hat{\Lambda}$ are the associated GNS maps. The maps \mathcal{F} and \mathcal{F}^{-1} are actually defined in larger subspaces, but we restricted the domains here

to D and \hat{D} , for convenience. As in the classical case, the Fourier inversion theorem holds:

$$\mathcal{F}^{-1}(\mathcal{F}(a)) = a, \quad a \in D, \quad \text{and} \quad \mathcal{F}(\mathcal{F}^{-1}(b)) = b, \quad b \in \hat{D}.$$

See [8] for more careful discussion on all these, including the definition of the Fourier transform and the proof of the result on the dual pairing.

3. THE QUANTUM DOUBLE

The quantum double construction was originally introduced by Drinfeld [6], in the Hopf algebra framework. The notion can be extended to the setting of locally compact quantum groups. See [23] (also see [7], and some earlier results in [15] and Section 8 of [2]). Some different formulations exist, but all of them are special cases of a more generalized notion of a *double crossed product construction* developed recently by Baaj and Vaes [3]. While we do not plan to go to the full generality as in that paper, let us give here the definition adapted from [3].

Let (N, Δ_N) be a locally compact quantum group, and let W_N be its multiplicative unitary operator. Write $(M_1, \Delta_1) = (N, \Delta_N^{\text{cop}})$ and $(M_2, \Delta_2) = (\hat{N}, \hat{\Delta}_N)$. Suggested by Proposition 8.1 of [3], consider the operators K and \hat{K} on $\mathcal{H} \otimes \mathcal{H}$:

$$K = W_N(\hat{J}_1 \otimes J_2)W_N^*, \quad \hat{K} = W_N(J_1 \otimes \hat{J}_2)W_N^*,$$

where $J_1, \hat{J}_1, J_2, \hat{J}_2$ are the modular conjugations for $M_1, \hat{M}_1, M_2, \hat{M}_2$. In our case, we would actually have: $\hat{J}_1 = J_2$ and $\hat{J}_2 = J_1$. Next, following Notation 3.2 of [3], write:

$$Z = K\hat{K}(\hat{J}_1 J_1 \otimes \hat{J}_2 J_2).$$

Then on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, define the unitary operator:

$$W_m = (\Sigma V_1^* \Sigma)_{13} Z_{34}^* W_{2,24} Z_{34}, \quad (3.1)$$

where V_1 (right regular representation of M_1) and W_2 (left regular representation of M_2) are multiplicative unitary operators associated with M_1 and M_2 . By Proposition 3.5 and Theorem 5.3 of [3], the operator W_m is a multiplicative unitary operator, and it gives rise to a locally compact quantum group (M_m, Δ_m) . This is the “double crossed product” (in the sense of Baaj and Vaes [3]) of (M_1, Δ_1) and (M_2, Δ_2) , and is to be called in Definition 3.1 below as the dual of the quantum double.

Definition 3.1. Let (N, Δ_N) be a locally compact quantum group, with W_N (“left regular representation”) and V_N (“right regular representation”) being the associated multiplicative unitary operators. In addition, denote by $J_N, \hat{J}_N, S_N, \varphi_N, \dots$ the relevant structure maps.

Let $(M_1, \Delta_1) = (N, \Delta_N^{\text{cop}})$, with the multiplicative unitary $W_1 = \Sigma V_N^* \Sigma$. We have: $J_1 = J_N$ and $\hat{J}_1 = \hat{J}_N$. Also $V_1 = (\hat{J}_1 \otimes \hat{J}_1) \Sigma W_1^* \Sigma (\hat{J}_1 \otimes \hat{J}_1)$. Since $J_1^2 = \hat{J}_1^2 = I_{\mathcal{H}}$, it becomes: $V_1 = \Sigma W_N^* \Sigma$. Meanwhile, let $(M_2, \Delta_2) =$

$(\hat{N}, \hat{\Delta}_N)$, which is associated with $W_2 = \Sigma W_N^* \Sigma$. We have: $J_2 = \hat{J}_N$ and $\hat{J}_2 = J_N$. Using these ingredients, construct the multiplicative unitary operator $W_m \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$, as given in equation (3.1). Then:

(1) The *Drinfeld quantum double* is (N_D, Δ_D) , given by the multiplicative unitary operator $W_D = \Sigma_{13} \Sigma_{24} W_m^* \Sigma_{24} \Sigma_{13}$. That is,

$$N_D = \overline{\{(\text{id} \otimes \text{id} \otimes \Omega)(W_D) : \Omega \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})_*\}}^w (\subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})),$$

with the comultiplication $\Delta_D : N_D \rightarrow N_D \otimes N_D$, defined by $\Delta_D(x) := W_D^*(1 \otimes 1 \otimes x)W_D$, for $x \in N_D$.

(2) The dual of the quantum double is $(\widehat{N}_D, \widehat{\Delta}_D)$, determined by W_m . Namely,

$$\widehat{N}_D = \overline{\{(\text{id} \otimes \text{id} \otimes \Omega)(W_m) : \Omega \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})_*\}}^w (\subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})),$$

with the comultiplication $\widehat{\Delta}_D : \widehat{N}_D \rightarrow \widehat{N}_D \otimes \widehat{N}_D$, given by $\widehat{\Delta}_D(y) := W_m^*(1 \otimes 1 \otimes y)W_m$, for $y \in \widehat{N}_D$.

By Theorem 5.3 of [3], it is known that N_D and \widehat{N}_D are locally compact quantum groups, equipped with suitable Haar weights φ_D and $\widehat{\varphi}_D$.

Note here that we took the dual of W_m in (1) to define the quantum double, so that our definition is more consistent with the ones given in the purely algebraic settings. Because of this, our $(\widehat{N}_D, \widehat{\Delta}_D)$ is none other than (M_m, Δ_m) , as defined in [3] (see that paper for details).

While the Baaj and Vaes paper [3] discusses these in a more general setting, it is to be noted that the case of the quantum double of a locally compact quantum group is not explicitly studied there. To be able to carry out the computations we have in mind, we need some specific details on the actual structure of the quantum double and its dual. This will be done in what follows. Note that our setting here is still more general than the discussions given in [15], [2], [23], [7].

For convenience, we will just write from now on that $\Delta = \Delta_N$ and $W = W_N$. In our case, $V_1 = \hat{W} = \Sigma W^* \Sigma$ and also $W_2 = \hat{W}$, while $J = J_N = J_1 = \hat{J}_2$ and $\hat{J} = \hat{J}_N = \hat{J}_1 = J_2$. So we will have:

$$Z = K\hat{K}(\hat{J}_1 J_1 \otimes \hat{J}_2 J_2) = W(\hat{J}J \otimes \hat{J}J)W^*(\hat{J}J \otimes J\hat{J}) \quad (3.2)$$

$$W_m = (\Sigma V_1^* \Sigma)_{13} Z_{34}^* W_{2,24} Z_{34} = W_{13} Z_{34}^* \hat{W}_{24} Z_{34} \quad (3.3)$$

$$W_D = Z_{12}^* W_{24} Z_{12} \hat{W}_{13} \quad (3.4)$$

We may occasionally be working at the C^* -algebra level. In that case, we will consider (A, Δ) and $(\hat{A}, \hat{\Delta})$, and the quantum double will be written as (A_D, Δ_D) , and its dual $(\widehat{A}_D, \widehat{\Delta}_D)$. We just need to work with the same multiplicative unitary operators but replace the weak completions above to the norm completions.

Let us begin first with $(\widehat{N}_D, \widehat{\Delta}_D) = (M_m, \Delta_m)$. See [3] for details.

Proposition 3.2. *As a von Neumann algebra, we have: $\widehat{N_D} = N \otimes \hat{N}$, while the comultiplication $\widehat{\Delta_D} : \widehat{N_D} \rightarrow \widehat{N_D} \otimes \widehat{N_D}$ is characterized as follows:*

$$\widehat{\Delta_D} = (\text{id} \otimes \sigma \circ m \otimes \text{id})(\Delta_1^{\text{cop}} \otimes \Delta_2) = (\text{id} \otimes \sigma \circ m \otimes \text{id})(\Delta \otimes \hat{\Delta}).$$

Here $\sigma : N \otimes \hat{N} \rightarrow \hat{N} \otimes N$ is the flip map, and $m : N \otimes \hat{N} \rightarrow N \otimes \hat{N}$ is the twisting map defined by $m(z) = ZZ^*$.

Its C^* -algebraic counterpart is rather tricky to describe. In general, unless W_D is regular (in the sense of Baaj and Skandalis [2]), it may be possible that $\widehat{A_D} \neq A \otimes \hat{A}$. See discussion given in Section 9 of [3]. Meanwhile, the description of the comultiplication $\widehat{\Delta_D}$ given above enables us to prove the following Lemma, which will be useful later:

Lemma 3.3. *Let $W = W_N$, $\hat{W} = \Sigma W^* \Sigma$, Z be the operators defined earlier. Then we have:*

$$Z_{34}Z_{12}^*W_{24}Z_{12}\hat{W}_{13} = \hat{W}_{13}Z_{12}^*W_{24}Z_{12}Z_{34}.$$

Proof. Since $W_m \in N_D \otimes \widehat{N_D}$ is the multiplicative unitary operator giving rise to the comultiplication $\widehat{\Delta_D}$, we should have (see [2]):

$$(\widehat{\Delta_D} \otimes \text{id})(W_m) = W_{m,13}W_{m,23}. \quad (3.5)$$

From the definition of W_m given in equation (3.3), the right side becomes:

$$W_{m,13}W_{m,23} = W_{15}Z_{56}^*\hat{W}_{26}Z_{56}W_{35}Z_{56}^*\hat{W}_{46}Z_{56}.$$

Meanwhile, remembering that $\hat{\Delta}(b) = \hat{W}^*(1 \otimes b)\hat{W}$ (for $b \in \hat{A}$) and that $\Delta(a) = W^*(1 \otimes a)W$ (for $a \in A$), we have:

$$\begin{aligned} (\Delta \otimes \hat{\Delta} \otimes \text{id})(W_m) &= (\Delta \otimes \hat{\Delta} \otimes \text{id})[W_{13}Z_{34}^*\hat{W}_{24}Z_{34}] \\ &= [W_{12}^*W_{25}W_{12}]Z_{56}^*[\hat{W}_{34}^*\hat{W}_{46}\hat{W}_{34}]Z_{56} \\ &= W_{15}W_{25}Z_{56}^*\hat{W}_{36}\hat{W}_{46}Z_{56} \\ &= W_{15}W_{25}[Z_{56}^*\hat{W}_{36}Z_{56}][Z_{56}^*\hat{W}_{46}Z_{56}]. \end{aligned}$$

In the third equality, we used the pentagon relations for W and for \hat{W} (being multiplicative unitaries). So we have:

$$\begin{aligned} (\widehat{\Delta_D} \otimes \text{id})(W_m) &= ((\text{id} \otimes \sigma \circ m \otimes \text{id})(\Delta \otimes \hat{\Delta}))(W_m) \\ &= Z_{32}W_{15}W_{35}[Z_{56}^*\hat{W}_{26}Z_{56}][Z_{56}^*\hat{W}_{46}Z_{56}]Z_{32}^* \\ &= W_{15}Z_{32}W_{35}[Z_{56}^*\hat{W}_{26}Z_{56}]Z_{32}^*[Z_{56}^*\hat{W}_{46}Z_{56}]. \end{aligned}$$

Therefore, the equation (3.5) now becomes (after obvious cancellations and then multiplying Z_{32}^* to both sides):

$$W_{35}[Z_{56}^*\hat{W}_{26}Z_{56}]Z_{32}^* = Z_{32}^*Z_{56}^*\hat{W}_{26}Z_{56}W_{35}.$$

Re-numbering the legs (legs 2,3,5,6 to become 4,3,1,2), we have:

$$W_{31}Z_{12}^*\hat{W}_{42}Z_{12}Z_{34}^* = Z_{34}^*Z_{12}^*\hat{W}_{42}Z_{12}W_{31}.$$

Now taking the adjoints from both sides, it becomes:

$$Z_{34}Z_{12}^* \hat{W}_{42}^* Z_{12}W_{31}^* = W_{31}^* Z_{12}^* \hat{W}_{42}^* Z_{12}Z_{34}.$$

Since $\hat{W} = \Sigma W^* \Sigma$, the result of Lemma follows immediately. \square

Let us now turn our attention to (N_D, Δ_D) . We will give a more concrete realization of N_D (in Proposition 3.4), as well as its coalgebra structure (in Proposition 3.5). See also Theorem 5.3 of [3].

Proposition 3.4. *Define $\pi : N \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ and $\pi' : \hat{N} \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ by*

$$\pi(f) := Z^*(1 \otimes f)Z \quad \text{and} \quad \pi'(k) := k \otimes 1.$$

Then N_D is the von Neumann algebra generated by the operators $\pi(f)\pi'(k)$, for $f \in N$, $k \in \hat{N}$. The maps π and π' are in fact W^ -algebra homomorphisms. Namely,*

$$\pi : N \rightarrow N_D \quad \text{and} \quad \pi' : \hat{N} \rightarrow N_D.$$

Proof. Recall from equation (3.4) that $W_D = Z_{12}^* W_{24} Z_{12} \hat{W}_{13}$. So for $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$, we have:

$$\begin{aligned} (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_D) &= (\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(Z_{12}^* W_{24} Z_{12} \hat{W}_{13}) \\ &= Z^* [1 \otimes (\text{id} \otimes \omega')(W)] Z [(\text{id} \otimes \omega)(\hat{W}) \otimes 1] = \pi(f)\pi'(k), \end{aligned}$$

where $f = (\text{id} \otimes \omega')(W)$ and $k = (\text{id} \otimes \omega)(\hat{W})$. This makes sense, because $W \in N \otimes \hat{N}$ and $\hat{W} \in \hat{N} \otimes N$. Recall the discussion in Section 2 above or Proposition 2.15 of [11]. In fact, the operators $(\text{id} \otimes \omega')(W)$, $\omega' \in \mathcal{B}(\mathcal{H})_*$, generate the von Neumann algebra N ; while the operators $(\text{id} \otimes \omega)(\hat{W})$, $\omega \in \mathcal{B}(\mathcal{H})_*$, generate \hat{N} .

Since the operators $(\text{id} \otimes \text{id} \otimes \omega \otimes \omega')(W_D)$ generate N_D by Definition 3.1, the claim of the proposition is proved. The second part of the proposition is obvious from the definitions. \square

Remark. For future computation purposes, we will from now on regard N_D to be the von Neumann algebra generated by the operators $\pi'(k)\pi(f)$, (for $f \in N$, $k \in \hat{N}$). This is of course true, given the results of the previous proposition. To be more specific, write:

$$\Pi(k \otimes f) := \pi'(k)\pi(f), \quad f \in N, \quad k \in \hat{N}. \quad (3.6)$$

Then we have: $N_D = \overline{\{\Pi(k \otimes f) : f \in N, k \in \hat{N}\}}^w$. Its C^* -algebraic counterpart is: $A_D = \overline{\{\Pi(k \otimes f) : f \in A, k \in \hat{A}\}}^{\parallel \parallel}$.

Proposition 3.5. *For $f \in N$ and $k \in \hat{N}$, we have:*

$$\begin{aligned} \Delta_D(\Pi(k \otimes f)) &= \Delta_D(\pi'(k)\pi(f)) = [(\pi' \otimes \pi')(\hat{\Delta}k)] [(\pi \otimes \pi)(\Delta f)] \\ &= (\Pi \otimes \Pi) \left(\sum k_{(1)} \otimes f_{(1)} \otimes k_{(2)} \otimes f_{(2)} \right). \end{aligned}$$

Proof. In the second line, we used the Sweedler's notation (see [14]), where we write: $\Delta f = \sum f_{(1)} \otimes f_{(2)}$. For computation, observe that

$$\begin{aligned}\Delta_D(\pi'(k)\pi(f)) &= W_D^*(1 \otimes 1 \otimes \pi'(k)\pi(f))W_D \\ &= [W_D^*(1 \otimes 1 \otimes \pi'(k))W_D][W_D^*(1 \otimes 1 \otimes \pi(f))W_D].\end{aligned}$$

Remembering the definitions of W_D and π' and π , we have:

$$\begin{aligned}W_D^*(1 \otimes 1 \otimes \pi'(k))W_D &= \hat{W}_{13}^*Z_{12}^*W_{24}^*Z_{12}(1 \otimes 1 \otimes k \otimes 1)Z_{12}^*W_{24}Z_{12}\hat{W}_{13} \\ &= \hat{W}_{13}^*(1 \otimes 1 \otimes k \otimes 1)\hat{W}_{13} = [\hat{\Delta}(k)]_{13} = (\pi' \otimes \pi')(\hat{\Delta}k).\end{aligned}$$

Meanwhile, by Lemma 3.3, we have:

$$\begin{aligned}W_D^*(1 \otimes 1 \otimes \pi(f))W_D &= \hat{W}_{13}^*Z_{12}^*W_{24}^*Z_{12}[Z^*(1 \otimes f)Z]_{34}Z_{12}^*W_{24}Z_{12}\hat{W}_{13} \\ &= Z_{34}^*Z_{12}^*W_{24}^*Z_{12}\hat{W}_{13}^*(1 \otimes 1 \otimes 1 \otimes f)\hat{W}_{13}Z_{12}^*W_{24}Z_{12}Z_{34} \\ &= Z_{34}^*Z_{12}^*W_{24}^*(1 \otimes 1 \otimes 1 \otimes f)W_{24}Z_{12}Z_{34} \\ &= Z_{34}^*Z_{12}^*[\Delta(f)]_{24}Z_{12}Z_{34} = (\pi \otimes \pi)(\Delta f).\end{aligned}$$

Combining these two results, we prove the proposition. \square

Remark. From the proof above, we see clearly that $(\pi \otimes \pi) \circ \Delta = \Delta_D \circ \pi$, and that $(\pi' \otimes \pi') \circ \hat{\Delta} = \Delta_D \circ \pi'$. From these observations, we see that the $*$ -homomorphisms π and π' defined earlier are also coalgebra homomorphisms.

As noted in Definition 3.1, the general theory assures us that (N_D, Δ_D) and $(\widehat{N_D}, \widehat{\Delta_D})$ are indeed (mutually dual) locally compact quantum groups. In particular, one can consider the (left) Haar weight φ_D of N_D and the (left) Haar weight $\widehat{\varphi_D}$ of $\widehat{N_D}$. We give the descriptions of φ_D and $\widehat{\varphi_D}$ below.

Proposition 3.6. (1) *The left Haar weight, φ_D , on (N_D, Δ_D) is characterized by the following:*

$$\varphi_D(\Pi(k \otimes f)) = \varphi_D(\pi'(k)\pi(f)) = \hat{\varphi}(k)\varphi(f), \quad \text{for } f \in N, k \in \hat{N}.$$

(2) *The left Haar weight, $\widehat{\varphi_D}$, on $\widehat{N_D} = N \otimes \hat{N}$ is as follows:*

$$\widehat{\varphi_D}(a \otimes b) = \varphi(a)\hat{\psi}(b).$$

Proof. For (2), concerning the Haar weight on $(\widehat{N_D}, \widehat{\Delta_D})$, see Theorem 5.3 of [3], which says: $\varphi_m = \psi_1 \otimes (\varphi_2)_{k_2}$. In our case, $\psi_1 = \varphi$, because $(M_1, \Delta_1) = (N, \Delta^{\text{cop}})$, while $\varphi_2 = \hat{\varphi}$, because $(M_2, \Delta_2) = (\hat{N}, \hat{\Delta})$. Moreover, our case being the ordinary quantum double of a locally compact quantum group, Proposition 8.1 of [3] indicates that $k_2 = \delta_2$, the “modular element” of $(\hat{N}, \hat{\Delta})$. We thus have: $(\varphi_2)_{k_2} = \hat{\varphi}\delta_2 = \hat{\psi}$.

Consider now φ_D given in (1). To verify the left invariance, recall Proposition 3.5 and compute:

$$\begin{aligned} (\Omega \otimes \varphi_D)(\Delta_D(\Pi(k \otimes f))) &= \sum (\Omega \otimes \varphi_D)((\Pi \otimes \Pi)(k_{(1)} \otimes f_{(1)} \otimes k_{(2)} \otimes f_{(2)})) \\ &= \sum [\Omega(\pi'(k_{(1)})\pi(f_{(1)}))\varphi_D(\pi'(k_{(2)})\pi(a_{(2)}))] \\ &= \sum [\Omega((k_{(1)} \otimes 1)Z^*(1 \otimes f_{(1)})Z)\hat{\varphi}(k_{(2)})\varphi(a_{(2)})]. \end{aligned}$$

Remembering the left invariance property of φ , which says: $\varphi((\omega \otimes \text{id})(\Delta f)) = \sum [\omega(f_{(1)})\varphi(f_{(2)})] = \omega(1)\varphi(f)$, and similarly for $\hat{\varphi}$, we thus have:

$$(\Omega \otimes \varphi_D)(\Delta_D(\Pi(k \otimes f))) = \Omega(1 \otimes 1)\hat{\varphi}(k)\varphi(f) = \Omega(1 \otimes 1)\varphi_D(\Pi(k \otimes f)),$$

which is none other than the left invariance property for φ_D . Though our proof is done only at the dense subalgebra level consisting of the $\Pi(k \otimes f)$, it is sufficient, since we already know the existence of the unique Haar weight from the general theory. By uniqueness, φ_D described here must be the dual Haar weight on (N_D, Δ_D) corresponding to $\widehat{\varphi_D}$. \square

Since we are not going to be prominently using them in this paper, we will skip the discussions on the right Haar weights and the antipode maps. But let us just remind the reader that the antipode map S_D can be obtained using the characterization given in equation (2.1), and similarly for $\widehat{S_D}$, working now with the operator W_D instead.

4. THE TWISTING OF THE QUANTUM DOUBLE

As is the case in the purely algebraic setting of QUE algebras [6], [4], the quantum double (A_D, Δ_D) or (N_D, Δ_D) is equipped with a “quantum universal R -matrix” type operator \mathcal{R} . Our plan is to use this operator to “twist (deform)” the comultiplication Δ_D .

Let us begin by giving the definition and the construction of \mathcal{R} , in the operator algebra setting. The approach is more or less the same as in Section 6 of [7], which was in turn adopted from Section 8 of [2]. On the other hand, some modifications were necessary, because the current situation is more general than those in [2] and in [7], where the discussions were restricted to the case of so-called “Kac systems”. At present, the proof here seems to be the one that is being formulated in the most general setting.

Lemma 4.1. *Let W, \hat{W}, Z be the operators in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ defined earlier. Then we have:*

- (1) $Z_{12}^* W_{45} W_{25} Z_{12} \hat{W}_{14} = \hat{W}_{14} Z_{12}^* W_{25} W_{45} Z_{12}$
- (2) $\hat{W}_{35} \hat{W}_{15} Z_{34}^* \hat{W}_{14} Z_{34} = Z_{34}^* \hat{W}_{14} Z_{34} \hat{W}_{15} \hat{W}_{35}$

Proof. Recall from Lemma 3.3 that $Z_{34} Z_{12}^* W_{24} Z_{12} \hat{W}_{13} = \hat{W}_{13} Z_{12}^* W_{24} Z_{12} Z_{34}$ or $Z_{12}^* Z_{34} W_{24} Z_{12} \hat{W}_{13} = \hat{W}_{13} Z_{12}^* W_{24} Z_{34} Z_{12}$. Recall now the definition of the operator Z given in equation (3.2), and write: $Z = WT$, where $T =$

$(\hat{J}J \otimes \hat{J}J)W^*(\hat{J}J \otimes J\hat{J})$. Then our equation above becomes (by writing $Z_{34} = W_{34}T_{34}$):

$$Z_{12}^*W_{34}T_{34}W_{24}Z_{12}\hat{W}_{13} = \hat{W}_{13}Z_{12}^*W_{24}W_{34}T_{34}Z_{12}.$$

We can cancel out T_{34} from both sides, because T_{34} actually commutes with all the operators in the equation. To see this, note that $\hat{J}J = kJ\hat{J}$, for a constant k (actually $k = \nu^{1/4}$, where $\nu > 0$ is the “scaling constant” [10], [11]). So we can write:

$$T = k(J \otimes \hat{J})(\hat{J} \otimes J)W^*(\hat{J} \otimes J)(J \otimes \hat{J}).$$

Since $W^* \in M \otimes \hat{M}$, we can see that $T \in M' \otimes \hat{M}'$.

So far we have: $Z_{12}^*W_{34}W_{24}Z_{12}\hat{W}_{13} = \hat{W}_{13}Z_{12}^*W_{24}W_{34}Z_{12}$. By re-numbering the legs (letting 3,4 to become 4,5), we obtain (1):

$$Z_{12}^*W_{45}W_{25}Z_{12}\hat{W}_{14} = \hat{W}_{14}Z_{12}^*W_{25}W_{45}Z_{12}.$$

Next, we re-write (1), using $\hat{W} = \Sigma W^* \Sigma$. Then we have:

$$Z_{12}^*\hat{W}_{54}^*\hat{W}_{52}^*Z_{12}\hat{W}_{14} = \hat{W}_{14}Z_{12}^*\hat{W}_{52}^*\hat{W}_{54}^*Z_{12}.$$

Apply $Z_{12}^*\hat{W}_{52}\hat{W}_{54}Z_{12}[\dots]Z_{12}^*\hat{W}_{54}\hat{W}_{52}Z_{12}$ to both sides. Then:

$$\hat{W}_{14}Z_{12}^*\hat{W}_{54}\hat{W}_{52}Z_{12} = Z_{12}^*\hat{W}_{52}\hat{W}_{54}Z_{12}\hat{W}_{14},$$

which is same as: $\hat{W}_{14}\hat{W}_{54}Z_{12}^*\hat{W}_{52}Z_{12} = Z_{12}^*\hat{W}_{52}Z_{12}\hat{W}_{54}\hat{W}_{14}$. Here, we re-number the legs (letting 1,2,4,5 to become 3,4,5,1), and obtain (2):

$$\hat{W}_{35}\hat{W}_{15}Z_{34}^*\hat{W}_{14}Z_{34} = Z_{34}^*\hat{W}_{14}Z_{34}\hat{W}_{15}\hat{W}_{35}.$$

□

Lemma 4.1 above will be helpful in our proof of the next proposition, which gives the description of our “quantum R -matrix” type operator \mathcal{R} .

Proposition 4.2. *Let $\mathcal{R} \in \mathcal{B}((\mathcal{H} \otimes \mathcal{H}) \otimes (\mathcal{H} \otimes \mathcal{H}))$ be the operator defined by $\mathcal{R} = Z_{34}^*\hat{W}_{14}Z_{34}$. The following properties hold:*

- (1) $\mathcal{R} \in M(A_D \otimes A_D) \subseteq N_D \otimes N_D$ and \mathcal{R} is unitary: $\mathcal{R}^{-1} = \mathcal{R}^*$.
- (2) We have: $(\Delta_D \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}$ and $(\text{id} \otimes \Delta_D)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}$.
- (3) For any $x \in A_D$, we have: $\mathcal{R}(\Delta_D(x))\mathcal{R}^* = \Delta_D^{\text{cop}}(x)$.
- (4) The operator \mathcal{R} satisfies the “quantum Yang-Baxter equation (QYBE)”: Namely, $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$.

Proof. Here $M(B)$ denotes the multiplier algebra of a C^* -algebra B .

(1) Recall that $\hat{W} \in \hat{N} \otimes N$. Therefore, by naturally extending the $*$ -homomorphisms π and π' defined in Proposition 3.4, we can see that $\mathcal{R} = (\pi' \otimes \pi)(\hat{W}) \in N_D \otimes N_D$. Actually, noting that $\hat{W} \in M(\hat{A} \otimes A)$, we also see that $\mathcal{R} \in M(A_D \otimes A_D)$. Meanwhile, from the definitions of the operators involved, it is clear that \mathcal{R} is unitary.

(2) Since $\mathcal{R} = (\pi' \otimes \pi)(\hat{W})$, we have:

$$\begin{aligned} (\Delta_D \otimes \text{id})(\mathcal{R}) &= (\Delta_D \otimes \text{id})((\pi' \otimes \pi)(\hat{W})) = (\pi' \otimes \pi' \otimes \pi)((\hat{\Delta} \otimes \text{id})(\hat{W})) \\ &= (\pi' \otimes \pi' \otimes \pi)(\hat{W}_{12}^* \hat{W}_{23} \hat{W}_{12}) = (\pi' \otimes \pi' \otimes \pi)(\hat{W}_{13} \hat{W}_{23}) \\ &= [(\pi' \otimes \pi' \otimes \pi)(\hat{W}_{13})] [(\pi' \otimes \pi' \otimes \pi)(\hat{W}_{23})] = \mathcal{R}_{13} \mathcal{R}_{23}. \end{aligned}$$

The second equality is due to $\Delta_D \circ \pi' = (\pi' \otimes \pi') \circ \hat{\Delta}$ (see Proposition 3.5). Since $\hat{W} \in M(\hat{A} \otimes A)$, the third equality follows from the definition of $\hat{\Delta}$. The fourth equality is the pentagon equation for \hat{W} (being multiplicative). The fifth equality is just using the fact that π' and π are C^* -homomorphisms.

The proof for $(\text{id} \otimes \Delta_D)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}$ can be done in a similar way. Just use the fact that for $a \in A$, we have: $\Delta a = W^*(1 \otimes a)W = \hat{W}_{21}(1 \otimes a)\hat{W}_{21}^*$, and that $\Delta_D \circ \pi = (\pi \otimes \pi) \circ \Delta$.

(3) Recall from Section 2 that $(\text{id} \otimes \omega)(\hat{W}) \in \hat{A}$, and $(\text{id} \otimes \omega')(W) \in A$, for $\omega, \omega' \in \mathcal{B}(\mathcal{H})_*$, and that these operators generate \hat{A} and A , respectively. So consider $b = (\text{id} \otimes \omega)(\hat{W}) \in \hat{A}$ and compute. Then:

$$\begin{aligned} \mathcal{R}[\Delta_D(\pi'(b))] &= \mathcal{R}[(\pi' \otimes \pi')(\hat{\Delta}b)] = (Z_{34}^* \hat{W}_{14} Z_{34})[\hat{W}_{13}^*(1 \otimes 1 \otimes b \otimes 1)\hat{W}_{13}] \\ &= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \omega)(Z_{34}^* \hat{W}_{14} Z_{34} \hat{W}_{13}^* \hat{W}_{35} \hat{W}_{13}) \\ &= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \omega)(Z_{34}^* \hat{W}_{14} Z_{34} \hat{W}_{15} \hat{W}_{35}) \\ &= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \omega)(\hat{W}_{35} \hat{W}_{15} Z_{34}^* \hat{W}_{14} Z_{34}) \\ &= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \omega)(\hat{W}_{31}^* \hat{W}_{15} \hat{W}_{31} Z_{34}^* \hat{W}_{14} Z_{34}) \\ &= [\hat{\Delta}^{\text{cop}}(b)]_{13} (Z_{34}^* \hat{W}_{14} Z_{34}) \\ &= [(\pi' \otimes \pi')(\hat{\Delta}^{\text{cop}}(b))] \mathcal{R} = [\Delta_D^{\text{cop}}(\pi'(b))] \mathcal{R}. \end{aligned}$$

The fourth and sixth equalities follow from the multiplicativity of \hat{W} , while the fifth equality is using Lemma 4.1 (2). In the seventh equality, we used the fact that $\hat{\Delta}^{\text{cop}}(b) = W(b \otimes 1)W^* = \hat{W}_{21}^*(b \otimes 1)\hat{W}_{21}$.

Next, consider $a = (\text{id} \otimes \omega')(W) \in A$ and compute. Then:

$$\begin{aligned} \mathcal{R}[\Delta_D(\pi(a))] &= (Z_{34}^* \hat{W}_{14} Z_{34})[Z_{12}^* Z_{34}^* W_{24}^*(1 \otimes 1 \otimes 1 \otimes a)W_{24} Z_{12} Z_{34}] \\ &= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \omega)(Z_{34}^* \hat{W}_{14} Z_{12}^* W_{24}^* W_{45} W_{24} Z_{12} Z_{34}) \\ &= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \omega)(Z_{34}^* \hat{W}_{14} Z_{12}^* W_{25} W_{45} Z_{12} Z_{34}) \\ &= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \omega)(Z_{34}^* Z_{12}^* W_{45} W_{25} Z_{12} \hat{W}_{14} Z_{34}) \\ &= (\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \omega)(Z_{34}^* Z_{12}^* W_{42}^* W_{25} W_{42} Z_{12} Z_{34} Z_{34}^* \hat{W}_{14} Z_{34}) \\ &= Z_{34}^* Z_{12}^* [\hat{W}_{24}(1 \otimes a \otimes 1 \otimes 1)\hat{W}_{24}^*] Z_{12} Z_{34} (Z_{34}^* \hat{W}_{14} Z_{34}) \\ &= [(\pi \otimes \pi)(\Delta^{\text{cop}}(a))] \mathcal{R} = [\Delta_D^{\text{cop}}(\pi(a))] \mathcal{R}. \end{aligned}$$

This is essentially the same computation as the previous one. The pentagon equation for W is used in the fourth and sixth equalities. The fifth equality is

using Lemma 4.1 (1). In the seventh and eighth equalities, note $\hat{W} = \Sigma W^* \Sigma$ and also note $\Delta^{\text{cop}}(a) = \Sigma W^*(1 \otimes a)W\Sigma = \hat{W}(a \otimes 1)\hat{W}^*$.

Since it has been observed that A_D is generated by the operators $\pi'(b)\pi(a)$, we conclude from the two results above (as well as the unitarity of \mathcal{R}) that: $\mathcal{R}[\Delta_D(x)]\mathcal{R}^* = \Delta_D^{\text{cop}}(x)$, for any $x \in A_D$.

(4) The QYBE follows right away from (3) and (4). In fact,

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{12}[(\Delta_D \otimes \text{id})(\mathcal{R})] = [(\Delta_D^{\text{cop}} \otimes \text{id})(\mathcal{R})]\mathcal{R}_{12} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

The first equality follows from (2); the second equality is from (3); and the third equality is from (2), with the legs 1 and 2 interchanged. \square

As a quick consequence of Proposition 4.2, we point out that \mathcal{R} determines a certain “left 2-cocycle” (dual to the notion of same name in the Hopf algebra setting, introduced in Section 3 of [12]). While we do not need to give the definition of a 2-cocycle here, this means that we can *deform* (or twist) the comultiplication Δ_D by multiplying \mathcal{R} from the left, and obtain a new map satisfying the coassociativity. The result is given below:

Proposition 4.3. *Let ${}_{\mathcal{R}}\Delta : A_D \rightarrow M(A_D \otimes A_D)$ be defined by*

$${}_{\mathcal{R}}\Delta(x) := \mathcal{R}\Delta_D(x), \quad \text{for } x \in A_D.$$

Then ${}_{\mathcal{R}}\Delta$ satisfies the coassociativity: $({}_{\mathcal{R}}\Delta \otimes \text{id}){}_{\mathcal{R}}\Delta = (\text{id} \otimes {}_{\mathcal{R}}\Delta){}_{\mathcal{R}}\Delta$.

Proof. The definition for ${}_{\mathcal{R}}\Delta$ makes sense, since $\mathcal{R} \in M(A_D \otimes A_D)$. Now for any $x \in A_D$, we have:

$$\begin{aligned} ({}_{\mathcal{R}}\Delta \otimes \text{id}){}_{\mathcal{R}}\Delta(x) &= \mathcal{R}_{12}(\Delta_D \otimes \text{id})(\mathcal{R}\Delta_D(x)) \\ &= \mathcal{R}_{12}[(\Delta_D \otimes \text{id})(\mathcal{R})][(\Delta_D \otimes \text{id})(\Delta_D(x))] \\ &= \mathcal{R}_{12}[\mathcal{R}_{13}\mathcal{R}_{23}][(\Delta_D \otimes \text{id})(\Delta_D(x))] \\ &= \mathcal{R}_{23}[\mathcal{R}_{13}\mathcal{R}_{12}][(\text{id} \otimes \Delta_D)(\Delta_D(x))] \\ &= \mathcal{R}_{23}[(\text{id} \otimes \Delta_D)(\mathcal{R})][(\text{id} \otimes \Delta_D)(\Delta_D(x))] \\ &= \mathcal{R}_{23}(\text{id} \otimes \Delta_D)(\mathcal{R}\Delta_D(x)) = (\text{id} \otimes {}_{\mathcal{R}}\Delta){}_{\mathcal{R}}\Delta(x). \end{aligned}$$

In the second and sixth equalities, we used the fact that Δ_D is a C^* -homomorphism. The third and fifth equalities used Proposition 4.2 (2). In the fourth equality, we used the QYBE and the coassociativity of Δ_D . \square

The coassociative map ${}_{\mathcal{R}}\Delta$ above is certainly a “deformed Δ_D ”. However, it should be noted that $(A_D, {}_{\mathcal{R}}\Delta)$ is not going to give us any valid quantum group. For instance, it is impossible to define a suitable Haar weight. And, ${}_{\mathcal{R}}\Delta$ is not even a * -homomorphism. On the other hand, considering that Δ_D is “dual” to the algebra structure on $\widehat{A_D}$ (via W_D and Proposition 2.2), and since ${}_{\mathcal{R}}\Delta$ still carries a sort of a “non-degeneracy” (since Δ_D is a non-degenerate C^* -morphism and \mathcal{R} is a unitary map), we may try to deform the algebra structure on $\widehat{A_D}$ by dualizing ${}_{\mathcal{R}}\Delta$. Formally, we wish to define

on the vector space $\widehat{A_D}$ a new product $\times_{\mathcal{R}}$, given by

$$\langle f \times_{\mathcal{R}} g | x \rangle = \langle f \otimes g | {}_{\mathcal{R}}\Delta(x) \rangle, \quad (4.1)$$

where $f, g \in \widehat{A_D}$ and $x \in A_D$.

The obvious trouble with this program is that $(A_D, {}_{\mathcal{R}}\Delta)$ is no longer a quantum group, which means that we do not have any multiplicative unitary operator that was essential in formulating the dual pairing in the case of locally compact quantum groups. In the next two sections, we will try to make sense of the formal equation (4.1), and use it to construct a C^* -algebra (though not a quantum group) that can be considered as a “deformed $\widehat{A_D}$ ”. Let us begin with the case of $A = C_{red}^*(G)$.

5. THE CASE OF AN ORDINARY GROUP. THE WEYL ALGEBRA.

For this section, let G be an ordinary locally compact group, with a fixed left Haar measure dx . Let $\nabla(x)$ denote the modular function. Using the Haar measure, we can form the Hilbert space $\mathcal{H} = L^2(G)$. We then construct two natural subalgebras, N and \hat{N} of $\mathcal{B}(\mathcal{H})$, as follows.

First consider the von Neumann algebra $N = \mathcal{L}(G)$, given by the left regular representation. That is, for $a \in C_c(G)$, let $L_a \in \mathcal{B}(\mathcal{H})$ be such that $L_a \xi(t) = \int a(z) \xi(z^{-1}t) dz$. We take $\mathcal{L}(G)$ to be the W^* -closure of $L(C_c(G))$. Next consider $\hat{N} = L^\infty(G)$, where $b \in L^\infty(G)$ is viewed as the multiplication operator μ_b on $\mathcal{H} = L^2(G)$, by $\mu_b \xi(t) = b(t) \xi(t)$. These are well-known von Neumann algebras, and it is also known that we can give (mutually dual) quantum group structures on them. We briefly review the results below.

Let $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) = \mathcal{B}(L^2(G \times G))$ be defined by $W\xi(s, t) = \xi(ts, t)$. It is actually the dual (that is, $W = \Sigma W_G^* \Sigma$) of the well-known multiplicative unitary operator W_G , defined by $W_G \xi(s, t) = \xi(s, s^{-1}t)$, and is therefore multiplicative [2]. We can show without difficulty that

$$N = \mathcal{L}(G) = \overline{\{(\text{id} \otimes \omega)(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w,$$

and the comultiplication on N is given by $\Delta(x) = W^*(1 \otimes x)W$, for $x \in N$. For $a \in C_c(G)$, this reads: $(L \otimes L)_{\Delta a} \xi(s, t) = \int a(z) \xi(z^{-1}s, z^{-1}t) dz$. The antipode map $S : a \rightarrow S(a)$ is such that $(S(a))(t) = \nabla(t^{-1})a(t^{-1})$, where ∇ is the modular function. The left Haar weight is given by $\varphi(a) = a(1)$, where $1 = 1_G$ is the group identity element. In this way, we obtain a von Neumann algebraic quantum group (N, Δ) , which is co-commutative.

Meanwhile, we can also show that:

$$\hat{N} = L^\infty(G) = \overline{\{(\omega \otimes \text{id})(W) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^w,$$

and the comultiplication on \hat{N} is given by $\hat{\Delta}(y) = \Sigma W(y \otimes 1)W^* \Sigma$, for $y \in \hat{N}$. In effect, this will give us $\hat{\Delta}b(s, t) = b(st)$, for $b \in L^\infty(G)$. The antipode map $\hat{S} : b \rightarrow \hat{S}(b)$ is such that $(\hat{S}(b))(t) = b(t^{-1})$, while the left Haar weight is just $\hat{\varphi}(b) = \int b(t) dt$. In this way, $(\hat{N}, \hat{\Delta})$ becomes a commutative von Neumann algebraic quantum group.

By considering the norm completions instead, we will have the C^* -algebraic quantum groups $A = C_{red}^*(G)$ and $\hat{A} = C_0(G)$. Meanwhile, as in Proposition 2.2, a dual pairing map can be considered at the level of certain dense subalgebras. For convenience, let us consider $L(C_c(G)) \subseteq N$ and $\mu(C_c(G)) \subseteq \hat{N}$. The dual pairing defined by the multiplicative unitary operator W , as given in equation (2.3) (or see Theorem 2.3), becomes:

$$\langle \mu_b | L_a \rangle = \int a(t)b(t^{-1}) dt, \quad (5.1)$$

for $\mu_b \in \mu(C_c(G))$ and $L_a \in L(C_c(G))$. The proof is straightforward.

We now turn to find a more concrete description of the quantum double, $D(G) = A_D$, and its dual $\widehat{D(G)}$. First, consider the operators J and \hat{J} on $\mathcal{H} = L^2(G)$, which come from our knowledge of the involution and the antipode maps.

$$J\xi(s) = \overline{\nabla(s^{-1})\xi(s^{-1})}, \quad \hat{J}\xi(s) = \overline{\xi(s)}.$$

Following the definitions given in Section 3, given by equations (3.2), (3.3), (3.4), construct the operator $Z \in \mathcal{B}(L^2(G \times G))$, as well as W_m and W_D , which act on $L^2(G \times G \times G \times G)$. We have:

$$\begin{aligned} Z\xi(s, t) &= W(\hat{J}J \otimes \hat{J}J)W^*(\hat{J}J \otimes J\hat{J})\xi(s, t) = \nabla(t^{-1})\xi(tst^{-1}, t). \\ W_m\xi(s, t, s', t') &= W_{13}Z_{34}^*\hat{W}_{24}Z_{34}\xi(s, t, s', t') = \nabla(t)\xi(s's, t, t^{-1}s't, t^{-1}t'). \\ W_D\xi(s, t, s', t') &= \Sigma_{13}\Sigma_{24}W_m^*\Sigma_{24}\Sigma_{13}\xi(s, t, s', t') \\ &= \nabla(t'^{-1})\xi(t'st'^{-1}, t't, t's^{-1}t'^{-1}s', t'). \end{aligned}$$

Next, by using the results of Propositions 3.2, 3.4, 3.5, we can give below the descriptions for the quantum double and its dual:

Proposition 5.1. *Let $A = C_{red}^*(G)$ and $\hat{A} = C_0(G)$ be the (mutually dual) quantum groups associated with G , equipped with their natural structure maps described above. Then:*

(1) *As a C^* -algebra, we have:*

$$D(G) = \overline{\{\Pi(\mu_k \otimes L_f) : f, k \in C_c(G)\}} \cong C_0(G) \rtimes_\alpha G,$$

where α is the conjugation action.

(2) *The comultiplication on $D(G)$ is given by*

$$\Delta_D(\Pi(\mu_k \otimes L_f)) = [(\pi' \otimes \pi')(\hat{\Delta}(\mu_k))] [(\pi \otimes \pi)(\Delta(L_f))].$$

(3) *As a C^* -algebra, we have: $\widehat{D(G)} = A \otimes \hat{A} = C_{red}^*(G) \otimes C_0(G)$.*

(4) *The comultiplication on $\widehat{D(G)}$ is given by*

$$\widehat{\Delta_D} = (\text{id} \otimes \sigma \circ m \otimes \text{id})(\Delta \otimes \hat{\Delta}),$$

where $m(z) = ZzZ^$, for $z \in M(A \otimes \hat{A})$.*

Proof. Recall equation (3.6) for the definition of Π , given in terms of the $*$ -homomorphisms π' and π from Proposition 3.4. For (1), note that:

$$\begin{aligned}\Pi(\mu_k \otimes L_f)\xi(s, t) &= \pi'(\mu_k)\pi(L_f)\xi(s, t) = (\mu_k \otimes 1)Z^*(1 \otimes L_f)Z\xi(s, t) \\ &= \int \nabla(z)k(s)f(z)\xi(z^{-1}sz, z^{-1}t) dz.\end{aligned}\quad (5.2)$$

If we write $\alpha_z\xi(s) = \xi(z^{-1}sz)$, $z \in G$, as the conjugation action, we can see without much difficulty from above that the C^* -algebra $D(G)$, which is generated by the operators $\Pi(\mu_k \otimes L_f)$, is isomorphic to the crossed product algebra $C_0(G) \rtimes_\alpha G$. [See any standard textbook on C^* -algebras, which contains discussion on crossed products.] By Proposition 3.5, we also know that the comultiplication on $D(G)$ is given as in (2).

In our case, being “regular”, we do have: $\widehat{D(G)} = A \otimes \hat{A}$. At the level of the functions in $C_c(G \times G)$, the multiplication on $\widehat{D(G)} = C_{red}^*(G) \otimes C_0(G)$ noted in (3) is reflected as follows:

$$[(a \otimes b) \times (a' \otimes b')](s, t) = \int a(z)b(t)a'(z^{-1}s)b'(t) dsdt. \quad (5.3)$$

The description given in (4) of the comultiplication $\widehat{\Delta_D}$ follows from Proposition 3.2. \square

The next proposition describes the dual pairing map. We may use equation (2.3), but we instead give our proof using Theorem 2.3.

Proposition 5.2. *The dual pairing map is defined between the (dense) subalgebras $(L \otimes \mu)(C_c(G \times G)) \subseteq \widehat{D(G)}$ and $\Pi((\mu \otimes L)(C_c(G \times G))) \subseteq D(G)$. Applying Theorem 2.3, we have:*

$$\begin{aligned}\langle L_a \otimes \mu_b | \Pi(\mu_k \otimes L_f) \rangle &= (\varphi_D \otimes \widehat{\varphi_D})[(\Pi(\mu_k \otimes L_f) \otimes 1 \otimes 1)W_D^*(1 \otimes 1 \otimes L_a \otimes \mu_b)] \\ &= \int \nabla(t)a(t^{-1}st)b(t^{-1})k(s)f(t) dsdt,\end{aligned}$$

where $L_a, L_f \in L(C_c(G)) \subseteq A$ and $\mu_b, \mu_k \in \mu(C_c(G)) \subseteq \hat{A}$.

Proof. Recall from Proposition 3.6 that the Haar weights φ_D and $\widehat{\varphi_D}$ are given by

$$\begin{aligned}\varphi_D(\Pi(\mu_k \otimes L_f)) &= \widehat{\varphi}(\mu_k)\varphi(L_f) = \int k(s)f(1) ds, \\ \widehat{\varphi_D}(L_a \otimes \mu_b) &= \varphi(L_a)\widehat{\psi}(\mu_b) = \varphi(L_a)\widehat{\varphi}(\hat{S}(\mu_b)) = \int a(1)b(t^{-1}) dt.\end{aligned}$$

Meanwhile, remembering the definitions of Π and W_D , we have:

$$\begin{aligned}(\Pi(\mu_k \otimes L_f) \otimes 1 \otimes 1)W_D^*(1 \otimes 1 \otimes L_a \otimes \mu_b)\xi(s, t, s', t') \\ = \int \nabla(z)\nabla(t')k(s)f(z)a(z')b(t')\xi(t'^{-1}z^{-1}szt', t'^{-1}z^{-1}t, z'^{-1}z^{-1}szs', t') dz dz'.\end{aligned}$$

By change of variables (first $z' \mapsto z^{-1}szz'$, and then $z \mapsto zt'^{-1}$), it becomes:

$$\begin{aligned} \dots &= \int \nabla(zt'^{-1})k(s)f(zt'^{-1})a(t'z^{-1}szt'^{-1}z')b(t')\xi(z^{-1}sz, z^{-1}t, z'^{-1}s', t') dz dz' \\ &= \int \nabla(z)F(s, z, s', t')\xi(z^{-1}sz, z^{-1}t, z'^{-1}s', t') dz dz' \\ &= ([\Pi \otimes (L \otimes \mu)](F))\xi(s, t, s', t'), \end{aligned}$$

where $F(s, z; z', t') = \nabla(t'^{-1})k(s)f(zt'^{-1})a(t'z^{-1}szt'^{-1}z')b(t') \in C_c(G \times G \times G \times G)$. Recall equation (5.2). Therefore,

$$\begin{aligned} \langle L_a \otimes \mu_b | \Pi(\mu_k \otimes L_f) \rangle &= (\varphi_D \otimes \widehat{\varphi_D})([\Pi \otimes (L \otimes \mu)](F)) \\ &= \int F(s, 1, 1, t^{-1}) ds dt = \int \nabla(t)k(s)f(t)a(t^{-1}st)b(t^{-1}) ds dt. \end{aligned}$$

□

By Theorem 2.3, we know that this is a valid dual pairing map (at the level of dense subalgebras) between $\widehat{D(G)}$ and $D(G)$, satisfying (1),(2),(3) of Proposition 2.2. In particular, the property (1) implies that:

$$\langle (L_a \otimes \mu_b)(L_{a'} \otimes \mu_{b'}) | \Pi(\mu_k \otimes L_f) \rangle = \langle (L_a \otimes \mu_b) \otimes (L_{a'} \otimes \mu_{b'}) | \Delta_D(\Pi(\mu_k \otimes L_f)) \rangle,$$

which relates the comultiplication Δ_D on $D(G)$ with the product on $\widehat{D(G)}$.

Even though we expressed our dual pairing as between certain subalgebras of $\widehat{D(G)}$ and $D(G)$, note that the pairing map is in effect being considered at the level of functions in $C_c(G \times G)$. In that sense, we may write the pairing map given in Proposition 5.2 as:

$$\langle a \otimes b | k \otimes f \rangle = \int \nabla(t)a(t^{-1}st)b(t^{-1})k(s)f(t) ds dt. \quad (5.4)$$

Let us now consider the deformed comultiplication $\mathcal{R}\Delta$ proposed in the previous section, and by using the dual pairing, try to “deform” the algebra $C^*(G) \otimes C_0(G)$. Since the dual pairing is valid only at the level of functions, we will first work in the subspace $C_c(G \times G)$. Formally, we wish to deform its product given in equation (5.3) to a new one, so that the new product is “dual” to $\mathcal{R}\Delta$, as suggested by equation (4.1). In our case, we look for the “deformed product” $\times_{\mathcal{R}}$, satisfying (formally) the following:

$$\langle [(a \otimes b) \times_{\mathcal{R}} (a' \otimes b')] | k \otimes f \rangle = \langle (a \otimes b) \otimes (a' \otimes b') | \mathcal{R}\Delta(k \otimes f) \rangle.$$

To make some sense of this, we first need to regard $\mathcal{R}\Delta(k \otimes f)$ as a (generalized) function on $G \times G$. So consider $k, f \in C_c(G)$, and consider $\Pi(\mu_k \otimes L_f) \in D(G)$. By definition, and by remembering that $\mathcal{R} = Z_{34}^* \hat{W}_{14} Z_{34}$, we have:

$$\begin{aligned} \mathcal{R}\Delta(\Pi(\mu_k \otimes L_f))\xi(s, t, s', t') &= \mathcal{R}\Delta_D(\Pi(\mu_k \otimes L_f))\xi(s, t, s', t') \\ &= \nabla(s)W_D^*(1 \otimes 1 \otimes \Pi(\mu_k \otimes L_f))W_D\xi(s, t, s^{-1}s's, s^{-1}t') \\ &= \int \nabla(s)\nabla(z)\nabla(z)k(s's)f(z)\xi(z^{-1}sz, z^{-1}t, z^{-1}s'sz, z^{-1}s^{-1}t') dz. \end{aligned}$$

Remembering the definition of Π as given in equation (5.2), we write it as:

$$\begin{aligned} \dots &= \int \nabla(z) \nabla(z') F(s, z; s', z') \xi(z^{-1}sz, z^{-1}t; z'^{-1}s'z', z'^{-1}t') dz dz' \\ &= [(\Pi \otimes \Pi)(F)] \xi(s, t; s', t'), \end{aligned}$$

where $F(s, z; s', z') = \nabla(s) \nabla(z) \nabla(z'^{-1}) k(s's) f(z) \delta_{z'}(sz)$. [Here, $\delta_{z'}(sz)$ is a “delta function”, such that for any function g , we have: $\int g(z') \delta_{z'}(sz) dz' = g(sz)$.] It is true that F is not really a function in $C_c(G \times G \times G \times G)$, but for our purposes, we may regard F as a (generalized) “function” expression corresponding to ${}_{\mathcal{R}}\Delta(\Pi(\mu_k \otimes L_f)) \in M(D(G) \otimes D(G))$.

Next, use equation (5.4) to compute the dual pairing (again formally). We then have:

$$\begin{aligned} \langle (a \otimes b) \otimes (a' \otimes b') | {}_{\mathcal{R}}\Delta(k \otimes f) \rangle &= \langle (a \otimes b) \otimes (a' \otimes b') | F \rangle \\ &= \int \nabla(t) \nabla(t') a(t^{-1}st) b(t^{-1}) a'(t'^{-1}s't') b'(t'^{-1}) F(s, t; s', t') ds dt ds' dt' \\ &= \int \nabla(t) \nabla(s) \nabla(t) a(t^{-1}st) b(t^{-1}) a'(t^{-1}s^{-1}s't) b'(t^{-1}s^{-1}) k(s's) f(t) ds dt ds'. \end{aligned}$$

By change of variables (letting $s' \mapsto s's^{-1}$ and then letting $s \mapsto tst^{-1}$), it becomes:

$$\begin{aligned} \dots &= \int \nabla(t) a(s) b(t^{-1}) a'(s^{-1}t^{-1}s't) b'(s^{-1}t^{-1}) k(s') f(t) ds dt ds' dt \\ &= \int \nabla(t) G(t^{-1}s't, t^{-1}) k(s') f(t) ds' dt = \langle G | k \otimes f \rangle, \end{aligned}$$

where $G(t^{-1}s't, t^{-1}) = \int a(s) b(t^{-1}) a'(s^{-1}t^{-1}s't) b'(s^{-1}t^{-1}) ds$. From which it follows that $G(p, t) = \int a(z) b(t) a'(z^{-1}p) b'(z^{-1}t) dz$.

Motivated by these computations (although not fully rigorous and depend on formal computations), we propose to define the “deformed product” $\times_{\mathcal{R}}$ on $C_c(G \times G)$, as follows:

$$[(a \otimes b) \times_{\mathcal{R}} (a' \otimes b')](s, t) = G(s, t) = \int a(z) b(t) a'(z^{-1}s) b'(z^{-1}t) dz.$$

Observe that $\times_{\mathcal{R}}$ is indeed a valid associative product on $C_c(G \times G)$. See below.

Proposition 5.3. *On $C_c(G \times G)$, define the “deformed product” $\times_{\mathcal{R}}$, as follows:*

$$[(a \otimes b) \times_{\mathcal{R}} (a' \otimes b')](s, t) = \int a(z) b(t) a'(z^{-1}s) b'(z^{-1}t) dz.$$

It is a valid associative product on $C_c(G \times G)$, and is “dual” to the deformed comultiplication ${}_{\mathcal{R}}\Delta$, in the (formal) sense described above.

Showing that $\times_{\mathcal{R}}$ is indeed an associative product on $C_c(G \times G)$ is quite straightforward. In fact, we can actually construct a C^* -algebra that contains $(C_c(G \times G), \times_{\mathcal{R}})$ as a dense subalgebra. The method is to follow the

standard procedure for constructing a crossed product C^* -algebra (where G acts on $C_0(G)$ by translation τ).

To be more specific, regard a typical element $a \otimes b \in C_c(G \times G)$ as an element $F \in C_c(G, C_0(G))$. We can then form the space $L^1(G, C_0(G))$, by completing $C_c(G, C_0(G))$ with respect to the following norm:

$$\|F\|_1 = \int_G \|F(s)\|_\infty ds = \int_G \sup_{t \in G} |F(s, t)| ds.$$

On this L^1 -space, we can consider the twisted convolution product and the adjoint operation, twisted by τ , obtaining the $*$ -algebra $L^1(G, C_0(G))$. Namely,

$$\begin{aligned} (F * G)(s) &= \int_G F(z) \tau_z(G(z^{-1}s)) dz, \\ F^*(s) &= \nabla(s^{-1}) \tau_s(F(s^{-1})^*). \end{aligned}$$

The crossed product C^* -algebra $C_0(G) \rtimes_\tau G$ is defined to be the enveloping C^* -algebra of the $*$ -algebra $L^1(G, C_0(G))$.

By viewing F and G as functions on $G \times G$, the multiplication and the $*$ -operation on the L^1 -algebra become:

$$\begin{aligned} (F * G)(s, t) &= \int_G F(z, t) G(z^{-1}s, z^{-1}t) dz, \\ F^*(s, t) &= \nabla(s^{-1}) \overline{F(s^{-1}, t)}. \end{aligned}$$

Observe that the twisted multiplication is none other than the deformed product $\times_{\mathcal{R}}$ given in Proposition 5.3. Therefore, the crossed product C^* -algebra $B = C_0(G) \rtimes_\tau G$ is a C^* -algebra containing $(C_c(G \times G), \times_{\mathcal{R}})$ as a dense subalgebra.

Proposition 5.4. *In view of the above discussion, we may regard the C^* -algebra $B = C_0(G) \rtimes_\tau G$ as a “deformed $\widehat{D(G)}$ ”, whose product is dual to the “deformed comultiplication” ${}_{\mathcal{R}}\Delta$ on $D(G)$. It contains $(C_c(G \times G), \times_{\mathcal{R}})$ as a dense subalgebra. Meanwhile, it is known that there exists an isomorphism of C^* -algebras between $C_0(G) \rtimes_\tau G$ (which is sometimes called the “Weyl algebra”) and the C^* -algebra of compact operators $\mathcal{K}(L^2(G))$. That is,*

$$C_0(G) \rtimes_\tau G \cong \mathcal{K}(L^2(G)).$$

As for the second characterization, see, for instance, [16]. By the way, note that in the von Neumann algebraic setting, our result would have been not much illuminating, since $\overline{\mathcal{K}(\mathcal{H})}^w = \mathcal{B}(\mathcal{H})$. This is the reason why we have chosen to work with the C^* -algebra framework in Sections 4 and 5.

Compare now with the finite-dimensional case, considered by Lu [12], [14]. Lu’s result says that given a Hopf algebra H , the twisting (via the R -matrix) of the dual of the quantum double turns out to be isomorphic to the “smash product” $H \# \hat{H}$, which is in turn isomorphic to $\text{End}(H)$ (see §9 of [14]). A similar result was obtained in [5], in the (also algebraic) setting

of multiplier Hopf algebras. Our result in Proposition 5.4 may be viewed as the C^* -algebraic counterpart to these results.

6. TOWARD THE GENERAL CASE.

Our program of finding a “twisted $\widehat{A_D}$ ” was successful in the ordinary group case, mainly because the dual pairing was simple to work with at the level of a nice subspace of continuous functions, namely $C_c(G) \subseteq A$. On the other hand, we know that the dual pairing is harder to work with in the general locally compact quantum group case. If we can reduce a little the role being played by the actual dual pairing formula, it is likely to lead us to an approach that is more general.

We believe that working with the generalized Fourier transform (as defined earlier) could be useful. In addition, while we wish to keep the overall strategy of the previous section, we also wish to find an approach that relies less on the existence of a dense subspace consisting of continuous functions. To find such an approach, let us first review the following fact.

Suppose that (M, Δ) is an arbitrary (von Neumann algebraic) locally compact quantum group, with its multiplicative unitary operator W . Recall from Section 2 that its dual object \hat{M} is given by

$$\hat{M} = \overline{\{(\omega \otimes \text{id})(W) : \omega \in M_*\}}^w.$$

What this means is that the von Neumann algebra \hat{M} is generated by the pre-dual M_* of M , via the “regular representation” $\lambda : \omega \mapsto (\omega \otimes \text{id})(W)$. Moreover, the operator multiplication makes M_* to be considered as an algebra. See Lemma 6.1 below:

Lemma 6.1. *Let (M, Δ) be a locally compact quantum group, with its multiplicative unitary operator W . Denote by M_* the pre-dual of the von Neumann algebra M . Then M_* can be given a natural algebra structure, together with a densely defined $*$ -operation:*

- (1) *For $\omega, \omega' \in M_*$, we have: $\lambda(\omega)\lambda(\omega') = \lambda(\mu)$ in \hat{M} , where $\mu \in M_*$ is such that*

$$\mu(x) = (\omega \otimes \omega')(\Delta x), \quad \text{for } x \in M.$$

- (2) *Write $\omega \in M_*^\sharp$, if $\omega \in M_*$ is such that there exists an element $\omega^\sharp \in M_*$, given by:*

$$\omega^\sharp(x) = \bar{\omega}(S(x)) = \overline{\omega([S(x)]^*)}, \quad \text{for all } x \in \mathcal{D}(S).$$

Then we have: $[\lambda(\omega)]^ = \lambda(\omega^\sharp)$ as operators in \hat{M} . Meanwhile, the subspace M_*^\sharp is a dense subalgebra (in the sense of (1) above) of M_* , which is closed under taking $^\sharp$.*

Remark. A similar result exist with roles of M and \hat{M} reversed. That is, we may think of the von Neumann algebra M being generated by the pre-dual \hat{M}_* of \hat{M} , via the “regular representation” $\hat{\lambda} : \theta \mapsto (\text{id} \otimes \theta)(\hat{W}) =$

$(\theta \otimes \text{id})(W^*)$. The situation is basically the same. All these are immediate consequences of the fact that the multiplicative unitary operator W associated with a locally compact quantum group is “manageable”. See the fundamental papers [2] and [22].

For $M = L^\infty(G)$, it is well-known that $M_* = L^1(G)$. So M_* , with its algebra structure given by Lemma 6.1, is a sort of an L^1 -algebra that generates the von Neumann algebra \hat{M} . This observation suggests that to “deform” \hat{M} (or \hat{A}), we may begin by deforming the algebra structure on M_* .

To follow this strategy in our case, consider now the quantum double (N_D, Δ_D) , whose C^* -algebraic counterpart is (A_D, Δ_D) . The multiplicative unitary operator is W_D , as defined in equation (3.4). To “deform” $\widehat{A_D}$, consider the pre-dual $(N_D)_*$ of the von Neumann algebra N_D , and let us introduce a new multiplication on it, as follows:

Proposition 6.2. *Let $(N_D)_*$ denote the pre-dual of the von Neumann algebra N_D . For $\omega, \omega' \in (N_D)_*$, define $\omega *_{\mathcal{R}} \omega' \in (N_D)_*$ by*

$$(\omega *_{\mathcal{R}} \omega')(x) := (\omega \otimes \omega')(\mathcal{R}\Delta_D(x)), \quad \text{for } x \in M_D.$$

*Then $*_{\mathcal{R}}$ is an associative multiplication on $(N_D)_*$.*

Proof. The associativity of $*_{\mathcal{R}}$ is an immediate consequence of the coassociativity of the map $x \mapsto \mathcal{R}\Delta_D(x) = \mathcal{R}\Delta(x)$, as noted in Proposition 4.3. \square

Let us now look for a representation Q of $((N_D)_*, *_{\mathcal{R}})$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. First, recall that the operators $(\omega \otimes \text{id})(W_D)$, $\omega \in (N_D)_*$, are dense in $\widehat{A_D}$. If we denote by $\hat{\Lambda}_D$ the GNS map for the Haar weight $\widehat{\varphi_D}$ of $\widehat{A_D}$, we thus know that the elements of the form $\hat{\Lambda}_D((\omega \otimes \text{id})(W_D))$ are dense in $\mathcal{H} \otimes \mathcal{H}$. This suggests the following definition of the “representation” Q . At the moment, no compatible $*$ -structure is specified on $((N_D)_*, *_{\mathcal{R}})$, so we only know that Q is an algebra homomorphism.

Definition 6.3. Define $Q : ((N_D)_*, *_{\mathcal{R}}) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ by

$$Q(\omega)\hat{\Lambda}_D((\nu \otimes \text{id})(W_D)) := \hat{\Lambda}_D([\omega *_{\mathcal{R}} \nu] \otimes \text{id})(W_D).$$

Since $*_{\mathcal{R}}$ is associative, and since the $\hat{\Lambda}_D((\nu \otimes \text{id})(W_D))$, $\nu \in (N_D)_*$, are dense in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$, this is certainly an algebra homomorphism, preserving the multiplication. Namely, $Q(\omega)Q(\omega') = Q(\omega *_{\mathcal{R}} \omega')$. Define B as the C^* -subalgebra of $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ generated by the $Q(\omega)$, $\omega \in (N_D)_*$. Then B may be considered as the “deformed $\widehat{A_D}$ ”.

Unfortunately, finding a more concrete description of the C^* -algebra B seems rather difficult. Recall that even before deforming, the C^* -algebra $\widehat{A_D}$ itself could be rather complicated in general. See comments following Proposition 3.2 and also see §9 of [3]. It is likely that the C^* -algebra B may be just as complicated.

In view of this obstacle, while we will try to push our strategy in the general case, we will soon restrict our attention to the case of $D(G)$, and

re-formulate the result of Section 5 using the new approach suggested by Definition 6.3. We hope that this alternative perspective can shed some light on the general case in the future.

With these remarks in mind, let us learn a little more about the subalgebra $Q((N_D)_*) \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. Suppose $\omega, \nu \in (N_D)_*$, and let $x \in N_D$ be arbitrary. Then by Definition 6.3, we have:

$$\begin{aligned} \langle Q(\omega)\hat{\Lambda}_D((\nu \otimes \text{id})(W_D)), \Lambda_D(x) \rangle &= \langle \hat{\Lambda}_D([\omega *_{\mathcal{R}} \nu] \otimes \text{id})(W_D), \Lambda_D(x) \rangle \\ &= (\omega *_{\mathcal{R}} \nu)(x^*) = (\omega \otimes \nu)(\mathcal{R}\Delta_D(x^*)) = (\omega \otimes \nu)(\Delta_D^{\text{cop}}(x^*)\mathcal{R}) \\ &= \langle \hat{\Lambda}_D((\omega \otimes \text{id})(W_D)) \otimes \hat{\Lambda}_D((\nu \otimes \text{id})(W_D)), (\Lambda_D \otimes \Lambda_D)(\mathcal{R}^*\Delta_D^{\text{cop}}(x)) \rangle. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product, the first two are on $\mathcal{H} \otimes \mathcal{H}$, while the last one is on $(\mathcal{H} \otimes \mathcal{H}) \otimes (\mathcal{H} \otimes \mathcal{H})$. The second and the fifth equalities are just using the definition of $\hat{\Lambda}_D$, as in equation (2.2). The third equality is from Proposition 6.2, and the fourth equality is the result of Proposition 4.2 (3).

Meanwhile, we know from Section 3 that $\widehat{N_D} = N \otimes \hat{N}$, which means that the elements $(\omega \otimes \text{id})(W_D)$, $\omega \in (N_D)_*$, are approximated by the elements of the form, $a \otimes b$, where $a \in \mathcal{A}(\subseteq N)$, $b \in \hat{\mathcal{A}}(\subseteq \hat{N})$. Therefore, the product $*_{\mathcal{R}}$ from Proposition 6.2 determines the “deformed product”, $\times_{\mathcal{R}}$, on a certain dense subspace of $N \otimes \hat{N}$. Then the computation above may be re-written as follows:

$$\begin{aligned} &\langle \hat{\Lambda}_D((a \otimes b) \times_{\mathcal{R}} (a' \otimes b')), \Lambda_D(x) \rangle \\ &= \langle \hat{\Lambda}_D(a \otimes b) \otimes \hat{\Lambda}_D(a' \otimes b'), \mathcal{R}^*(\Lambda_D \otimes \Lambda_D)(\Delta_D^{\text{cop}}(x)) \rangle \\ &= \langle \mathcal{R}[\hat{\Lambda}_D(a \otimes b) \otimes \hat{\Lambda}_D(a' \otimes b')], (\Lambda_D \otimes \Lambda_D)(\Delta_D^{\text{cop}}(x)) \rangle. \end{aligned} \quad (6.1)$$

Here we are using the fact $(\Lambda_D \otimes \Lambda_D)(\mathcal{R}^*\Delta_D^{\text{cop}}(x)) = \mathcal{R}^*[(\Lambda_D \otimes \Lambda_D)(\Delta_D^{\text{cop}}(x))]$, which is true since $\mathcal{R} \in N_D \otimes N_D$ and since the GNS representation associated with Λ_D is just the inclusion map $N_D \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.

Let us denote by \mathcal{F}_D and \mathcal{F}_D^{-1} the Fourier transform and the inverse Fourier transform between certain dense subalgebras of N_D and $\widehat{N_D}$, defined in the same way as in Theorem 2.3. By the property of the Fourier transform (see Propositions 3.5 and 3.7 of [8]), it is known that $\hat{\Lambda}_D(\mathcal{F}_D(x)) = \Lambda_D(x)$ and that $\Lambda_D(\mathcal{F}_D^{-1}(y)) = \hat{\Lambda}_D(y)$, where $x \in N_D$ and $y \in \widehat{N_D}$ are assumed to be contained in suitable domains. We thus have:

$$\begin{aligned} \mathcal{R}[\hat{\Lambda}_D(a \otimes b) \otimes \hat{\Lambda}_D(a' \otimes b')] &= \mathcal{R}[\Lambda_D(\mathcal{F}_D^{-1}(a \otimes b)) \otimes \Lambda_D(\mathcal{F}_D^{-1}(a' \otimes b'))] \\ &= (\Lambda_D \otimes \Lambda_D)(\mathcal{R}[\mathcal{F}_D^{-1}(a \otimes b) \otimes \mathcal{F}_D^{-1}(a' \otimes b')]) \\ &= (\hat{\Lambda}_D \otimes \hat{\Lambda}_D)[(\mathcal{F}_D \otimes \mathcal{F}_D)(\mathcal{R}[\mathcal{F}_D^{-1}(a \otimes b) \otimes \mathcal{F}_D^{-1}(a' \otimes b')])]. \end{aligned} \quad (6.2)$$

Remark. If we formally extend the Fourier transform, then by the Fourier inversion theorem, we may write $\mathcal{R} = (\mathcal{F}_D^{-1} \otimes \mathcal{F}_D^{-1})((\mathcal{F}_D \otimes \mathcal{F}_D)(\mathcal{R}))$. Then the expression in the last line above is essentially the “convolution product”,

as defined in Proposition 3.11 of [8]. That is,

$$(\mathcal{F}_D \otimes \mathcal{F}_D)(\mathcal{R}[\mathcal{F}_D^{-1}(a \otimes b) \otimes \mathcal{F}_D^{-1}(a' \otimes b')]) = (\mathcal{F}_D \otimes \mathcal{F}_D)(\mathcal{R} \ast ((a \otimes b) \otimes (a' \otimes b'))).$$

We may use the result in [8] to write down an alternative description for the convolution product, using the Haar weight and the antipode map.

Comparing our computations in this section with Proposition 2.2 (1), we see that

$$\begin{aligned} & (a \otimes b) \times_{\mathcal{R}} (a' \otimes b') \\ &= ((m_N)_{31} \otimes (m_{\hat{N}})_{42})[(\mathcal{F}_D \otimes \mathcal{F}_D)(\mathcal{R}[\mathcal{F}_D^{-1}(a \otimes b) \otimes \mathcal{F}_D^{-1}(a' \otimes b')])], \end{aligned} \quad (6.3)$$

where m_N and $m_{\hat{N}}$ denote the multiplications on N and \hat{N} , respectively.

While the formula given in equation (6.3) is not entirely rigorous, it does give us a workable description (assuming the details like the operator W_D , the Haar weights, and the Fourier transforms are known) of the “deformed product” $\times_{\mathcal{R}}$, on a dense subspace contained in $\mathcal{A} \otimes \hat{\mathcal{A}}$. This is essentially the multiplication on $(N_D)_*$ given in Proposition 6.2.

As we indicated earlier in the section, we do not plan to carry out the computations in full generality, which seems rather difficult. Instead, let us from now on return to the set up and the notations given in Section 5, corresponding to $N = \mathcal{L}(G)$ and $\hat{N} = L^\infty(G)$. As before, it is convenient to work with the space of functions having compact support.

Lemma 6.4. *Let $a, b \in C_c(G)$ and consider $L_a \otimes \mu_b \in N \otimes \hat{N} = \widehat{N_D}$. Then:*

$$\mathcal{F}_D^{-1}(L_a \otimes \mu_b) = (\mu_a \otimes 1)Z^*(1 \otimes L_{\tilde{b}})Z = \Pi(\mu_a \otimes L_{\tilde{b}}) \in N_D,$$

where $\tilde{b}(t) = \nabla(t^{-1})b(t)$. [Recall that ∇ is the modular function.]

Proof. By definition,

$$\mathcal{F}_D^{-1}(L_a \otimes \mu_b) = (\text{id} \otimes \widehat{\varphi_D})(W_D^*([1 \otimes 1] \otimes [L_a \otimes \mu_b])).$$

Since $\widehat{\varphi_D} = \varphi \otimes \hat{\psi}$ (see Proposition 3.6 (2)), this becomes:

$$\begin{aligned} \mathcal{F}_D^{-1}(L_a \otimes \mu_b) &= (\text{id} \otimes \text{id} \otimes \varphi \otimes \hat{\psi})(\hat{W}_{13}^* Z_{12}^* W_{24}^* Z_{12}(1 \otimes 1 \otimes L_a \otimes \mu_b)) \\ &= (\text{id} \otimes \text{id} \otimes \varphi \otimes \hat{\psi})([\hat{W}^*(1 \otimes L_a)]_{13} Z_{12}^* [W^*(1 \otimes \mu_b)]_{24} Z_{12}) \\ &= ([(\text{id} \otimes \varphi)(\hat{W}^*(1 \otimes L_a))] \otimes 1) Z^*(1 \otimes [(\text{id} \otimes \hat{\psi})(W^*(1 \otimes \mu_b))] Z). \end{aligned}$$

But remembering that $\hat{W} = \Sigma W^* \Sigma$, we have:

$$(\text{id} \otimes \varphi)(\hat{W}^*(1 \otimes L_a)) = (\varphi \otimes \text{id})(W(L_a \otimes 1)) = \mathcal{F}(L_a) = \mu_a,$$

where the last result was shown in Section 5 of [8], and can be obtained by a direct computation. Similarly, $(\text{id} \otimes \hat{\psi})(W^*(1 \otimes \mu_b)) = \mathcal{F}^{-1}(\mu_b) = L_b$. Since $\hat{\psi}$ and φ are related by the modular function (in general, related via the “modular operator”), we can show without much difficulty that

$$(\text{id} \otimes \hat{\psi})(W^*(1 \otimes \mu_b)) = L_{\tilde{b}},$$

where $\tilde{b} \in C_c(G)$ is as defined above.

Combining the results, we indeed have:

$$\mathcal{F}_D^{-1}(L_a \otimes \mu_b) = (\mu_a \otimes 1)Z^*(1 \otimes L_{\tilde{b}})Z = \Pi(\mu_a \otimes L_{\tilde{b}}).$$

□

While the above Lemma was formulated for the case of $N = \mathcal{L}(G)$ and $\hat{N} = L^\infty(G)$, we can see from the proof that a reasonable generalization (using the Fourier transform) could be given for more general settings. In this paper, we will be content with the current description, since we will be using a computational method in what follows.

Let us now put together the results so far. In our case, with the Fourier transform being rather simple (see Lemma 6.4), the actual computation is not too difficult. By a straightforward computation, the expression in equation (6.2) becomes:

$$(\mathcal{F}_D \otimes \mathcal{F}_D)(\mathcal{R}[\mathcal{F}_D^{-1}(L_a \otimes \mu_b) \otimes \mathcal{F}_D^{-1}(L_{a'} \otimes \mu_{b'})]) = (L \otimes \mu \otimes L \otimes \mu)(F),$$

where $F \in C_c(G \times G \times G \times G)$ is given by

$$F(s, t, s', t') = \nabla(s)a(s)b(t)a'(s^{-1}s')b'(s^{-1}t').$$

Next, equation (6.3) will provide us with the deformed product $\times_{\mathcal{R}}$ on $C_c(G \times G)$, as follows:

$$\begin{aligned} [(a \times b) \times_{\mathcal{R}} (a' \otimes b')](s, t) &= [((m_N)_{31} \otimes (m_{\hat{N}})_{42})(F)](s, t) \\ &= \int F(z^{-1}s, t, z, t) dz = \int \nabla(z^{-1}s)a(z^{-1}s)b(t)a'(s^{-1}z)z^{-1}s)b'(s^{-1}zt) dz \\ &= \int \nabla(s)a(zs)b(t)a'(s^{-1}z^{-1}s)b'(s^{-1}z^{-1}t) dz \\ &= \int a(z)b(t)a'(z^{-1}s)b'(z^{-1}t) dz. \end{aligned} \tag{6.4}$$

In the fourth and fifth equalities, we used the change of variables, $z \mapsto z^{-1}$, and then $z \mapsto zs^{-1}$.

Observe that we obtain the multiplication on $C_c(G \times G)$ that is exactly the same as the one given in Proposition 5.3. As we indicated earlier, this is none other than the deformed product on $(N_D)_*$ as in Proposition 6.2. Moreover, the C^* -algebra $B = C_0(G) \rtimes_{\tau} G$, which was shown in Section 5 to be the completion of $(C_c(G \times G), \times_{\mathcal{R}})$ will be the C^* -algebra generated by the $Q(\omega)$, $\omega \in (N_D)_*$, as described in Definition 6.3.

The computations here support our definition of the “deformed $\widehat{A_D}$ ” as given in Definition 6.3. It is an improvement, since the definition is given in a fairly general manner, and since a very straightforward way of construction is also obtained via equation (6.3).

However, we note that the last part of the process, realizing the product given in equation (6.3), needs further improvement. While the method is reasonably practical in the sense that once we have enough information

(about the Haar weight, the multiplicative unitary operator, and the Fourier transform) we can carry out the construction, it will be more desirable if we can reduce our dependence on specific computational results.

With this remark in mind, let us include the following observation, which may be relevant for future generalization of our program:

Proposition 6.5. *Let the notations be as above. Then:*

$$\begin{aligned} B = C_0(G) \rtimes_{\tau} G &= \overline{\{(1 \otimes \mu_b^{\text{op}})(\Delta(L_a)) : a, b \in C_c(G)\}}^{\parallel} \\ &= (1 \otimes \hat{A}^{\text{op}})\Delta(A) \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{H}). \end{aligned}$$

Remark. Here, \hat{A}^{op} is the C^* -algebra corresponding to \hat{N}' , equipped with the opposite multiplication, being denoted by μ^{op} . In our case, working with μ^{op} is just nominal, since the product on $\hat{N} = L^\infty(G)$ is already known to be commutative. We nevertheless chose to use μ^{op} , anticipating a possible future generalization. Indeed, the description above was obtained from some heuristic computations exploiting the close relationship between the multiplicative unitary operator \hat{W} and the operator $\mathcal{R} = Z_{34}^* \hat{W} Z_{14} Z_{34}$.

Proof. Let $a, b \in C_c(G)$ and let $\xi \in \mathcal{H} \otimes \mathcal{H}$. Then by the results obtained in Section 5, we have:

$$(1 \otimes \mu_b^{\text{op}})(\Delta(L_a))\xi(s, t) = \int b(t)a(z)\xi(z^{-1}s, z^{-1}t) dz.$$

Comparing this with the concrete realization we obtained in equation (6.4) for the product on the C^* -algebra B (see also Section 5), the result of the proposition follows. \square

Unless the quantum group (A, Δ) is “regular” (in the sense of Baaj and Skandalis [2], [22]), the C^* -algebra $(1 \otimes \hat{A})\Delta^{\text{cop}}(A)$ is not necessarily isomorphic to $\mathcal{K}(\mathcal{H})$ and in general may be quite complicated (It may not even be “type I”. See [18] and Section 9 of [3].). Meanwhile, even though we cannot provide a general proof here, several computations at the heuristic level (using different examples) seem to suggest that this is the correct description for the C^* -algebra B . We hope to report on this matter in the near future.

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